IMC 2020 Preparation Course

Lecture 2 - Algebra

University of Warwick

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1. Let $f(x) = x^{2012} + a_{2011}x^{2011} + ... + a_0$. Albert Einstein and Homer Simpson are playing a game. They alternate at assigning real values to the coefficients of $f(x)$. Homer's goal is to make $f(x)$ to be divisible by a fixed polynomial $m(x)$. Albert's goal is to prevent this. Which player has a winning strategy if a) $m(x) = x - 2012$ and if b) $m(x) = x^2 + 1$?

2. Prove that $n + 3$ and $n^2 + 3n + 3$ cannot both be cubes for any natural number n.

3. Find the minimum for $log_{x_1}(x_2 - \frac{1}{4}) + log_{x_2}(x_3 - \frac{1}{4}) + ... + log_{x_n}(x_1 - \frac{1}{4})$.

4. Find natural numbers $n, k_1, ..., k_n$ such that $k_1 + ... + k_n = 5n - 4$ and $\frac{1}{k_1} + ... + \frac{1}{k_n} = 1$.

5. Let A, B, C be commuting matrices with complex entries. Prove that there are real numbers α, β, γ which are not all equal to zero such that $det(\alpha A + \beta B + \gamma C) = 0$.

6. Let $S = \emptyset$. Players A and B choose elements of S_n and add them to S in alternate order. The first player whose move makes S a generating set for S_n loses. Which player has a winning strategy for each value of n?

7. Let a, b be two integers and suppose that n is a natural number such that the set $S = Z \setminus \{ax^n + by^n\}$ is finite. Prove that $n = 1$ and $S = \emptyset$.

8. Let A be a symmetric $m \times m$ matrix with entries from F_2 all of whose diagonal entries are zero. Prove that, for every natural number n, each column of the matrix A^n has a zero entry.

9. Consider $n+1$ points $A_1, ..., A_n$ in R^n, no k lying in the same $k-1$-dimensional subspace for $k < n+1$, and let B be strictly inside their convex hull. Show that there are at least n obtuse angles of the form A_iBA_j, $i < j$.

10. Let $inv(\pi)$ denote the number of inversions of a permutation π. Let $f(n)$ be the number of permutations $\pi \in S_n$ for which $n+1|inv(\pi)$. Prove that there are infinitely many primes p for which $f(p-1) > \frac{(p-1)!}{p}$, as well as infinitely many primes p for which $f(p-1) < \frac{(p-1)!}{p}$.