IMC 2020 Preparation Course

Lecture 1 - Analysis

University of Warwick

February 19, 2020

1. Let $0 < a < b$. Prove that $\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}$.

2. a) A sequence $(x_n)_{n\in N}$ satisfies the recurrence relation $x_{n+1} = x_n cos(x_n)$. Does it converge for every choice of x_1?
b) A sequence $(y_n)_{n\in N}$ satisfies the recurrence relation $y_{n+1} = y_n sin(y_n)$. Does it converge for every choice of y_1?

3. Consider the sequence $(a_n)_{n\in N}$ with $a_0 = 1, a_1 = \frac{1}{2}$ and $a_{n+1} = \frac{na_n^2}{1+(n+1)a_n}$ for $n > 1$. Show that the series $\Sigma_{n=1}^{\infty} \frac{a_{n+1}}{a_n}$ converges and determine its value.

4. Consider the sequence $(a_n)_{n\in N} = (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1...)$. Find all pairs of positive numbers (α, β) such that $lim_{n\to\infty} \frac{\Sigma_{k=1}^n a_k}{n^\alpha} = \beta$.

5. Find all functions $f : N \to N$ satisfying $f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001$.

6. Show that the sequence $\sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7+\sqrt{7}}}, \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}},...$ converges, and calculate its limit.

7. Let $f : R \to R$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in R$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

8. Let $P(x)$ be a polynomial with real coefficients such that, for every $n \in N$, the equation $P(x) = n$ has at least one rational root. Prove that $P(x) = ax + b$, where a, b are rational numbers.

9. Let $(f_n)_{n\in N} : [0,1) \to R$ be a sequence of continuously differentiable functions satisfying the recurrence $f_1 = 1$, $f'_{n+1} = f_n f_{n+1}$ on $(0,1)$, and $f_{n+1}(0) = 1 \forall n \in N$. Show that the pointwise limit of this sequence exists for every $x \in [0,1)$, and determine its value.

10. Let $Q(x)$ be a polynomial of degree $n \geq 1$ with integer coefficients. Prove that $max_{\{0 \leq x \leq 1\}} |p(x)| > \frac{1}{e^n}$.