IMC 2020 Preparation Course

Lecture 3 - Combinatorics

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1. Show that every graph has two vertices with the same degree.

2. Let $p(n)$ denote the number of ways that the positive integer n can be written as a sum of positive integers (convention: $p(0) = 1$). Show that the number of ways that n can be written as a sum of integers each of which is strictly greater than 1 is $p(n) - p(n-1)$.

3. Consider a 3-partite graph whose parts all have size n and whose each vertex has at least k neighbours in both of the parts in which it is not contained. Show that: i) if $k = \frac{n}{2}$, it is possible that there are no 3-cycles, ii) if $k \geq \frac{3n}{4}$, the vertices can be partitioned in disjoint 3-cycles.

4. Consider the sequence $a_n = \Sigma_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (-4)^{-k}$. Find rational numbers a, b, c such that $a_n = (an + b)c^n$ for every $n \in N$.

5. How many permutations of [2014] are there such that $|\sigma(1) - 1| = |\sigma(2) - 2| = ... = |\sigma(2014) - 2014| > 0$?

6. Let A be a 101-element subset of $S = [1000000]$. Prove that there are elements $t_1, ..., t_{100} \in S$ such that the sets $A + t_j$ are pairwise disjoint.

7. Given a set of at least $d + 1$ points in a d-dimensional space such that any d of them can be covered by a unit ball, prove that all of the points can be covered by a unit ball.

8. If $a_1, ..., a_n, b_1, ...b_n$ are distinct integers such that the sums $a_i + a_j$ are the same as the sums $b_i + b_j$ (in some order), prove that n is a power of 2.

9. Given a necklace with 2013 beads, we colour it white and green, and dub the colouring "good" if between any 21 consecutive beads there is one that is green. Prove that the number of good colourings is odd (a colouring is considered distinct from its dihedral images).

10. Let G be an abelian group and $A \subset G$ a finite set satisfying $|A + A| \leq c|A|$ for some $c \in R$. Prove that $|\Sigma_{i=1}^{k} A| \leq c^k |A|$ (Plunnecke's Inequality).