IMC Selection Test 1

- Work on the test between 12pm and 4pm UK time (**at most 4 hours**)
- Books, notes, calculators and internet **are not allowed**
- If you have questions about the test, you can call me via MS Teams 12–1pm (during the first hour)
- Make a separate pdf file for each attempted question, naming it `ID_Surname_Question.pdf` (for example `1234567_Smith_2.pdf`)
- Email your solutions (as pdf files) **before 5pm** to `o.pikhurko@warwick.ac.uk` with subject: IMC Test 1

Problem 1 Let k and n be positive integers. The n-th derivative of $\frac{1}{x^k-1}$ can be written in the form $\frac{P_{k,n}(x)}{(x^k-1)^{n+1}}$ where $P_{k,n}(x)$ is some polynomial. Find $P_{k,n}(1)$.

Problem 2 Recall that the Fibonacci sequence $(u_n)_{n=1}^{\infty}$ is defined by $u_1 := 1$, $u_2 := 1$ and $u_n := u_{n-1} + u_{n-2}$ for $n \geq 3$. Prove that there is an integer $n \geq 1$ such that $u_n \equiv -1 \pmod{2021^{2021}}$.

Problem 3 Let $S := \mathbb{Q} \setminus \{-1, 0, 1\}$ denote the set of rational numbers different from -1, 0 and 1. Define $f : S \to S$ by $f(x) := x - 1/x$ for $x \in S$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Problem 4 Let $k, n \geq 3$ be integers. Let B be a set of more than $(k-1)2^n/n$ distinct points of the form $(\pm 1, \ldots, \pm 1)$ (that is, each coordinate is -1 or 1) in the n-dimensional Euclidean space $\mathbb{R}^n$. Show that there are k distinct points in B forming a regular simplex (i.e., all $\binom{k}{2}$ distances between them are the same).

Problem 5 Let $g(z) := 11z^{10} + 10iz^9 + 10iz - 11$, where z is a complex number and $i^2 = -1$. Prove that if $g(z) = 0$ for $z \in \mathbb{C}$, then $|z| = 1$.