IMC Selection Test 2

- Work on the test between 12pm and 4pm UK time (**at most 4 hours**)
- Books, notes, calculators and internet **are not allowed**
- If you have questions about the test, you can call me via MS Teams 12–1pm (during the first hour of the test)
- Make a separate pdf file for each attempted question, naming it `ID_Surname_Question.pdf` (for example `1234567_Smith_2.pdf`), the same naming scheme as for Test 1
- Email your solutions (as pdf files) **before 5pm** to `o.pikhurko@warwick.ac.uk` with subject: IMC Test 2

Problem 1 Let $s_0 := 1$ and, inductively on $n = 1, 2, 3, \dots$, define $s_n := (\frac{6n+1}{n+1} + s_{n-1})^{1/3}$. Does the limit $\lim_{n\to\infty} s_n$ exist? If yes, determine its value.

Problem 2 Find $a, b, c \in \mathbb{R}$ such that for every closed convex polygon R in the plane it holds that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-D(x,y)}\,\mathrm{d}x\,\mathrm{d}y = a + bP + cA,$$

where $D(x, y)$ is the Euclidean distance from $(x, y) \in \mathbb{R}^2$ to the nearest point of R, P is the perimeter of R and A is the area of R.

Problem 3 An unbiased coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails? You have to find a closed formula for your answer.

Problem 4 Prove that $x \mapsto 2x$ is the unique function $f : (0, \infty) \to (0, \infty)$ such that

$$f(f(x)) = 6x - f(x), \quad \text{for all } x \in (0, \infty).$$

Problem 5 Let H be a finite group and let K be a subgroup of H. Show that there exist elements $h_1, h_2, \dots, h_n \in H$ with $n = |H|/|K|$, such that $h_1K, h_2K, ..., h_nK$ are the left cosets of K and $Kh_1, Kh_2, \dots, Kh_n$ are the right cosets of K (i.e. each of these two n-tuples forms a partition of H). Here, for $h \in H$, we denote $hK := \{hk : k \in K\}$ and $Kh := \{kh : k \in K\}$.