IMC Selection Test 1

Problem 1 Solution: By differentiating $P_{k,n}(x)/(x^k-1)^{n+1}$, we find that $P_{k,n+1}(x) = (x^k-1)P'_{k,n}(x) - (n+1)kx^{k-1}P_{k,n}(x)$; substituting $x = 1$ yields $P_{k,n+1}(1) = -(n+1)kP_{k,n}(1)$. Since $P_{k,0}(1) = 1$, an easy induction gives that $P_{k,n}(1) = (-k)^n n!$ for all $n \geq 0$. ∎

Problem 2 Solution: The Fibonacci recursion can be used to extend the sequence 'backwards', namely by defining $u_n := u_{n+2} - u_{n+1}$ for $n = 0, -1, -2, \dots$. Since there are finitely many possible values for the pair of residues $(u_n \pmod m, u_{n+1} \pmod m)$ where $m := 2021^{2021}$, some pair repeats. Clearly, any pair $(u_n \pmod m, u_{n+1} \pmod m)$ uniquely determines the residue modulo m of u_k for every $k \in \mathbb{Z}$. It follows that the residues of the extended sequence, namely $(u_n \pmod m)_{n\in\mathbb{Z}}$, are periodic. As $u_0 = 0$ and $u_{-1} = -1$, the value -1 appears in the periodic sequence of residues, in particular, $u_n \equiv -1 \pmod m$ for some $n \geq 1$. ∎

Problem 3 Solution: For a rational number p/q expressed in lowest terms, define its *height* $H(p/q)$ to be $|p| + |q|$. Then for any $p/q \in S$ expressed in lowest terms, we have

$$H(f(p/q)) = H((p^2-q^2)/(pq)) = |q^2-p^2| + |pq|,$$

since coprimality of p and q implies coprimality of $p^2 - q^2$ and pq. As p and q are nonzero integers with $|p| \neq |q|$, we have

$$\begin{aligned} H(f(p/q)) - H(p/q) &= |q^2-p^2| + |pq| - |p| - |q| \\ &\geq 3 + |pq| - |p| - |q| \\ &= (|p|-1)(|q|-1) + 2 \geq 2. \end{aligned}$$

It follows that $f^{(n)}(S)$ consists solely of numbers of height strictly larger than $2n+2$, and hence

$$\cap_{n=1}^{\infty} f^{(n)}(S) = \emptyset.$$

Note that many other choices for the height function are possible: one can take $H(p/q) := \max\{|p|, |q|\}$, or let $H(p/q)$ be equal to the total number of factors of p and q, and so on. The key properties of the height function are that on one hand, there are only finitely many rationals with height below any finite bound while, on the other hand, $H(a) < H(f(a))$ for every $a \in S$. ∎

Problem 4 Solution: Let S be the set of all 2^n points with coordinates $(\pm 1, \dots, \pm 1)$. For each $P \in S$, let S_P consists of those vectors in S that differ from P in exactly one coordinate. Let m be the number of pairs (P, Q) with $P \in B$ and $Q \in S_P$. Clearly, each S_P has exactly n elements. Hence, $m = n|B| > (k-1)2^n$. It follows that some element $Q \in S$ forms at least $m/|S| = n|B|/2^n > k-1$ pairs (P, Q). The corresponding k vectors P form the required simplex. Indeed, every two of these vectors differ in exactly 2 coordinates, so the side length is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. ∎

Problem 5 Solution: The equation $g(z) = 0$ can be re-written as

$$z^9 = \frac{11 - 10iz}{11z + 10i}.$$

If $z = a + bi$ with $a, b \in \mathbb{R}$, then

$$|z^9| = \left|\frac{11 - 10iz}{11z + 10i}\right| = \frac{\sqrt{11^2 + 220b + 10^2(a^2 + b^2)}}{\sqrt{11^2(a^2 + b^2) + 220b + 10^2}}.$$

Let $n(a, b)$ and $d(a, b)$ denote the numerator and the denominator of the right-hand side. If $|z| > 1$, then $a^2 + b^2 > 1$, so $d(a, b) > n(a, b)$, making $|z^9| < 1$, a contradiction. If $|z| < 1$, then $a^2 + b^2 < 1$, so $d(a, b) < n(a, b)$, making $|z^9| > 1$, again a contradiction. Hence $|z| = 1$, as required.

An alternative solution is as follows. Let $p(z) := -z^5 g(z)$ for $z \in \mathbb{C}$. One can show that $p(\mathrm{e}^{i\theta}) = 22\cos(5\theta) + 20\cos(4\theta)$ for $\theta \in \mathbb{R}$. As θ increases from 0 to 2π, the real-valued function $22\cos(5\theta) + 20\cos(4\theta)$ changes sign at least 10 times, since at $\theta = 2\pi k/10$ for $k = 0, 1, \ldots, 10$ its value is $22(-1)^k + 20\cos(4\theta)$, which has the sign of $(-1)^k$. By the Intermediate Value Theorem, the continuous function $p(\mathrm{e}^{i\theta})$ has at least 10 zeros on $[0, 2\pi)$. Thus $g(z)$ also has at least 10 zeros on the circle $|z| = 1$ and these are all the zeros since $g(x)$ is a polynomial of degree 10. ∎