IMC Selection Test 2

Problem 1 Solution: For convenience, denote $a_n := (6n+1)/(n+1)$. First we show by mathematical induction that the sequence (s_n) is non-decreasing. Note that $s_0 = 1 < s_1 = (9/2)^{1/3}$. Assume that $s_{n-1} \le s_n$. Note that $a_n < a_{n+1}$. Thus we have

$$s_n = (a_n + s_{n-1})^{1/3} \le (a_{n+1} + s_n)^{1/3} = s_{n+1}.$$

Next we show, also by induction, that the sequence (s_n) is bounded above by 2. Clearly, $s_0 = 1 < 2$. Assume that $s_{n-1} < 2$. Then we have

$$s_n = (a_n + s_{n-1})^{1/3} < (6+2)^{1/3} = 2.$$

Thus $L := \lim_{n\to\infty} s_n$ exists. Letting $n \to \infty$ in $s_n^3 = a_n + s_{n-1}$, we obtain $L^3 = 6 + L$. The roots are 2 and $-1 \pm i\sqrt{2}$. Since each s_n is a real number, the limit is 2. ∎

Problem 2 Solution: Let $I(S)$ denotes the integral of $\mathrm{e}^{-D(x,y)}$ over a region S. Since $D(x,y) = 0$ on R, it holds that $I[R] = A$.

For a subset σ of the boundary of R, let $S(\sigma)$ consists the points in the plane outside of R having a point on σ as the nearest point of R. (A nearest point exists since R is closed and is unique since R is convex.)

Let σ be a side of R (where we do not include its end-points) and let s be its length. Then $S(\sigma)$ is the half-strip of width s with σ as its base and

$$I(S(\sigma)) = \int_0^s \int_0^\infty \mathrm{e}^{-v}\, \mathrm{d}v\ \mathrm{d}u = s.$$

Adding this over all sides of R, we get exactly P.

For a vertex σ of angle $\pi - \alpha$, using polar coordinates we get that

$$I[S(\sigma)] = \int_0^\alpha \int_0^\infty re^{-r}\, \mathrm{d}\theta\ \mathrm{d}r = \alpha,$$

since $\int_0^\infty re^{-r}\, \mathrm{d}r = (r(-e^{-r}))|_{r=0}^\infty - \int_0^\infty (r)'e^{-r}\, \mathrm{d}r = 0 - (-1) = 1$. We have that the sum of $\alpha(\sigma)$ over all vertices σ of P is 2π: to see this either use the fact that the sum of the angles of an n-gon is $\pi(n-2)$ or see how the angles α add up, when we take line perpendiculars to the sides. Thus adding $I[S(\sigma)]$ over all vertices of R, we get 2π.

Finally, by the uniqueness of the nearest point, we have that the integral in question $I(\mathbb{R}^2)$ is $2\pi + P + A$. Thus the stated formula is true with $a = 2\pi$ and $b = c = 1$. ∎

Problem 3 Solution: For each $k \in \{0, \ldots, n\}$ the probability of $T = k$ is $2^{-n}\binom{n}{k}$ and the events that $T > H$ and $T < H$ are equally likely. Thus the expectation of $\frac{1}{2}|H - T|$

is equal to

$$\sum_{k=0}^{\lfloor n/2\rfloor}(n-2k)\binom{n}{k} = \sum_{k=0}^{\lfloor n/2\rfloor}\left((n-k)\binom{n}{k}-k\binom{n}{k}\right)$$
$$= n\sum_{k=0}^{\lfloor n/2\rfloor}\left(\binom{n-1}{k}-\binom{n-1}{k-1}\right) = n\binom{n-1}{\lfloor (n-1)/2\rfloor}.$$

Thus the expectation of $|H-T|$ is $\frac{n}{2^{n-1}}\binom{n-1}{\lfloor (n-1)/2\rfloor}$ ∎

Problem 4 Solution: Fix any $a_0 > 0$ and inductively let $a_{n+1} := f(a_n)$ for $n = 0, 1, 2, \dots$. Then each a_n is defined and $a_{n+2} = -a_{n+1}+6a_n$. The zeros of the characteristic equation $t^2 = -t + 6$ are 2 and -3 so there are $c_1, c_2 \in \mathbb{R}$ such that $a_n = c_1 2^n + c_2(-3)^n$ for all integers $n \geq 0$. Since each $a_n > 0$, it follows that $c_2 = 0$. Thus $a_{n+1} = 2a_n$ for every $n \geq 0$, in particular $f(a_0) = 2a_0$. Since $a_0 > 0$ was arbitrary we indeed have $f(x) = 2x$ for all $x \in (0, \infty)$. ∎

Problem 5 Solution: Let L (resp. R) be the set of left (resp. right) cosets of K. Let G be the bipartite graph on a disjoint union of L and K, where we connect xK to Ky if they intersect. If we take any $X \subseteq L$, then the cosets in X cover between them $|X| \cdot |K|$ elements of H and thus at least $|X|$ right cosets intersect X. Thus Hall's condition that the neighbourhood of any set X in one part of a bipartite graph has at least $|X|$ elements holds. By Hall's Marriage Theorem there is a perfect matching $M = \{e_1, \dots, e_n\}$. For each $e_i = \{x_iK, Ky_i\}$, let h_i be any element of $x_iK \cap Ky_i \neq \emptyset$. These elements h_i have the stated properties. ∎