IMC Selection Test 1

Problem 1 Supposee that the sum of the elements of every row of an invertible matrix A is s. Prove that the sum of the elements of every row of its inverse A^{-1} is $1/s$.

Problem 2 [Replace 1992 by 2022.] Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

Problem 3 Suppose that sets $A_1, \dots A_m$, each of size r, satisfy that $|A_i \cap A_j| \le k$ for all $1 \le i < j \le m$. Prove that

$$|A_1 \cup \dots \cup A_m| \ge \frac{m\,r^2}{k(m-1)+r}.$$

Problem 4 Let α_1 and α_2 be irrational numbers. Prove that there are infinitely many pairs of rationals $(p_1/q, p_2/q)$ such that $|\alpha_i - p_i/q| < q^{-3/2}$ for both $i = 1$ and $i = 2$.