IMC Selection Test 1

Solution to Problem 1. Plugging $x = 0$, we get $P(1) = 1$. Plugging $x = 1$, we get $P(2) = 2$, plugging $x = 2$, we get $P(5) = 5$, and so on. Thus $P(x)$ and x coincide in infinitely many values of x, so they are identical.

Solution to Problem 2. We can prove by induction that $A^kB - BA^k = kA^k$ for all $k \geq 1$. Clearly the base case $k = 1$ is given by assumption. Now assume $A^kB - BA^k = kA^k$ for some $k \geq 1$. Then

$$\begin{aligned} A^{k+1}B &= A(BA^k + kA^k) = ABA^k + kA^{k+1} \\ &= (BA + A)A^k + kA^{k+1} = BA^{k+1} + (k+1)A^{k+1} \end{aligned}$$

thus proving the induction step. This implies $\mathrm{tr}(kA^k) = \mathrm{tr}(A^kB - BA^k) = 0$, and so $\mathrm{tr}(A^k) = 0$ for all $k \geq 1$. Hence, A is nilpotent, and so $\det(A) = 0$.

Solution to Problem 3. For each $1 \leq i \leq n$, let B_i be the ball of radius $\frac{1}{4}$ centered at x_i. Then from assumption, these n balls are disjoint and are all contained in the ball of radius $\frac{5}{4}$ centered at the origin. By computing area, we have $\frac{\pi n}{16} \leq \frac{25\pi}{16}$, which implies that $n \leq 25$.

Solution to Problem 4. Taking $x = y = 0$ gives $f(f(0) = 0$. For brevity, we let $c := f(0)$. Now taking $x = 0$ gives $f(|f(y) - c|) = f(y^2)$, and taking $y = 0$ gives $f(|f(x) - c|) = f(f(x)) + c(1 - 2x^2)$. By combining these two equations, we get $f(f(x)) = f(x^2) + c(2x^2 - 1)$. Substituting back into (**??**) gives us $f(|f(x) - f(y)|) = f(x^2) + f(y^2) + c(2x^2 - 1) - 2x^2 f(y)$. As the left hand side is symmetrical in x and y, we get $c(x^2 - y^2) = x^2 f(y) - y^2 f(x)$. Now taking $y = 1$ shows that $f(x) = ax^2 + b$ for some $a, b \in \mathbb{R}$.

Substituting this into (**??**) gives $a^3(x^2 - y^2)^2 = f(f(x)) - 2x^2(ay^2 + b) + ay^4$. Now letting $x = 0$ gives $a^3y^4 = ay^4$ for all $y \in \mathbb{R}$, and thus $a \in \{-1, 0, 1\}$. Checking each of these possibilities gives the following solutions: $f \in \{0, x^2, x^2 - 1, -x^2, -x^2 + 1\}$.