IMC Geometry exercises

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1. (IMC 2006 Day 2 Q1) Let V be a convex polygon with n vertices.
   (a) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.
   (b) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.

2. (IMC 2009 Day 2 Q1) Let l be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to l is greater than or equal to two times the distance between X and P. If the distance from P to l is $d > 0$, find the volume of S.

3. (IMC 2003 Day 2 Q3) Let A be a closed subset of $\mathbb{R}^n$, and let B be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that
$$|a_0 - b| = \inf_{a \in A} |a - b|.$$
   Prove that B is dense in $\mathbb{R}^n$; that is, given a point $x \in \mathbb{R}^n$, there are points in B that are arbitrarily close to x.

4. Let n be a positive integer, and $k \geq 2$. Suppose $\{x_1, x_2, \ldots, x_k\}$ is a finite set of distinct points in $\mathbb{R}^n$. Prove that there is a hyperplane separating one of the x_i from the remaining $k-1$ points.

   See how many different solutions you can find to this problem!

5. (IMC 2004 Day 2 Q3) Let D be the closed unit disk in the plane, and let $p_1, p_2, \ldots, p_n$ be fixed points in D. Show that there exists a point p in D such that the sum of the distances of p to each of $p_1, p_2, \ldots, p_n$ is greater than or equal to n.

6. (IMC 2004 Day 1 Q4) Suppose $n \geq 4$ and let M be a finite set of n points in $\mathbb{R}^3$, no four of which lie in a plane. Assume that the points can be coloured black or white so that any sphere which intersects M in at least four points has the property that exactly half of the points in the intersection of M and the sphere are white. Prove that all of the points in M lie on one sphere.

7. (IMC 2008 Day 2 Q2) Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

8. (IMC 2008 Day 2 Q6) Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) \,:\, x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.

9. (IMC 2009 Day 1 Q5) Let n be a positive integer. An n-simplex in $\mathbb{R}^n$ is specified by $n + 1$ points $P_0, P_1, \ldots, P_n$, called its *vertices*, which do not all belong to the same hyperplane. For every n-simplex S denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S. Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)c(S).$$