IMC 2023 Training
Calculus

Problems

1. (IMC 2020 Problem 5) Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying
$$f''(x)f(x) \geq 2(f'(x))^2$$
for all $x \in \mathbb{R}$.

2. (IMC 2019 Problem 3) Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that
$$2f'(x) + xf''(x) \geq 1 \text{ for } x \in (-1, 1).$$
Prove that
$$\int_{-1}^{1} xf(x)dx \geq \frac{1}{3}.$$

3. (IMC 2019 Problem 6) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that
$$(f(0) - g'(0))(g'(1) - f(1)) > 0.$$
Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

4. (IMC 2018 Problem 4) Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that
$$f(b) - f(a) = (b - a)f'(\sqrt{ab}) \text{ for all } a, b > 0.$$

5. (Putnam 2015 B1) Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

6. (Putnam 1997 B2) Let f be a twice differentiable real valued function satisfying
$$f(x) + f''(x) = -xg(x)f'(x),$$
where $g(x) > 0$ for all real x. Prove that $|f(x)|$ is bounded.

7. (VJMC 2019 II P2) Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f''(x)\cos(f(x)) \geq (f'(x))^2 \sin(f(x)) \text{ for every } x \in \mathbb{R}.$$

8. (VJMC 2013 I P1) Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function with $|f(x)| \leq M$ and
$$f(x)f'(x) \geq \cos x \text{ for } x \in [0, \infty),$$
where $M > 0$. Prove that $f(x)$ does not have a limit as $x \to \infty$.

Exercise

1. (IMC2022 Problem 1) Let $f : [0,1] \to (0,\infty)$ be an integrable function such that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Prove that
$$\int_0^1 f(x)dx \geq 1.$$

2. (IMC2021 Problem 4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for any $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0,\infty)$ such that for every pair (x,y) of real numbers,
$$\text{if } |x-y| < \min\{g(x), g(y)\}, \text{ then } |f(x) - f(y)| < \varepsilon.$$
Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

3. (IMC2021 Problem 7) Let $D \subset \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that
$$|f(0)| \leq \max_{|z|=1} |f(z)p(z)|.$$

4. (Putnam 2018 A5) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exists a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

5. (Putnam 2017 A3) Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a,b]$ to $(0,\infty)$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$ but $f \neq g$. For every positive integer n, define
$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\, dx.$$
Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

6. (Putnam 2016 A3) Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that
$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$
for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find
$$\int_0^1 f(x)dx.$$

7. (VJMC 2017 I P4) Let $f : (1,\infty) \to \mathbb{R}$ be a continuously differentiable function satisfying $f(x) \leq x^2 \log(x)$ and $f'(x) > 0$ for every $x \in (1,\infty)$. Prove that
$$\int_1^\infty \frac{1}{f'(x)}\, dx = \infty.$$

8. (VJMC 2017 II P2) Prove or disprove the following statement. If $g : (0,1) \to (0,1)$ is an increasing function and satisfies $g(x) > x$ for all $x \in (0,1)$, then there exists a continuous function $f : (0,1) \to mathbbR$ satisfying $f(x) < f(g(x))$ for all $x \in (0,1)$, but f is not an increasing function.

9. (VJMC 2016 I P1) Let $f : \mathbb{R} \to (0,\infty)$ be a continuously differentiable function. Prove that there exists $\xi \in (0,1)$ such that

$$e^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}.$$

10. (VJMC 2016 II P4) Let $f : [0,\infty) \to \mathbb{R}$ be a continuously

11. (SMMC 2021 B3) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following two properties.

    (i) The Riemann integral $\displaystyle\int_a^b f(t)dt$ exists for all real numbers $a < b$.

    (ii) For every real number x and every integer $n \geq 1$ we have

$$f(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)dt.$$

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Solutions

Useful theorems

Theorem 1 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function such that*

- *f is continuous on $[a, b]$, and*
- *f is differentiable on (a, b), and*
- *$f(a) = f(b)$.*

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 2 (Lagrange's mean value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function such that*

- *f is continuous on $[a, b]$, and*
- *f is differentiable on (a, b).*

Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 3 (Darboux's theorem)**.** *Let $f(x)$ be a function such that $f'(x)$ exists for any $x \in [a, b]$. Then for any y between $f'(a)$ and $f'(b)$, there exists $\xi \in (a, b)$ such that $f'(\xi) = y$.*

Theorem 4 (Taylor's theorem)**.** *Let a and x be any real numbers. Let $f(x)$ be a continuous function such that the $n+1$-th derivative of $f(x)$ exists between a and x. Then there exists ξ between a and x such that*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x - a)^n}{n!} + \frac{f^{(n+1)}(\xi)(x - a)^{n+1}}{(n + 1)!}.$$

Solutions

1. (IMC 2020 Problem 5) Let $g(x) = \dfrac{1}{f(x)}$. We are going to prove that g is constant and hence f is constant. Observe that

$$\begin{aligned} g' &= -\frac{f'}{f^2} \\ g'' &= \frac{2(f')^2 - ff''}{f^3} \le 0. \end{aligned}$$

Suppose there exists $a \in \mathbb{R}$ such that $g'(a) \neq 0$. For any $x \in \mathbb{R}$, we can find ξ between a and x, by Taylor's theorem, which shows that

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(\xi)(x-a)^2}{2!} \le g(a) + g'(a)(x-a).$$

Then by taking $x = a - \dfrac{2g(a)}{g'(a)}$, we have

$$g(x) \le g(a) + g'(a)\left(\left(a - \frac{2g(a)}{g'(a)}\right) - a\right) = -g(a) < 0$$

which is impossible since $g(x) = \dfrac{1}{f(x)} > 0$ for any $x \in \mathbb{R}$. It follows that $g'(x) = 0$ for any $x \in \mathbb{R}$. Therefore $g(x)$ is a constant function which implies $f(x)$ is a constant function.

2. (IMC 2019 Problem 3) Let

$$g(x) = xf(x) - \frac{x^2}{2},$$

Observe that

$$\begin{aligned} g'(x) &= f(x) + xf'(x) - x \\ g''(x) &= 2f'(x) + xf''(x) - 1 \ge 0. \end{aligned}$$

For any $x \in (-1,1)$, we can find ξ between 0 and x, by Taylor's theorem, which shows that

$$g(x) = g(0) + g'(0)x + \frac{g''(\xi)x^2}{2} \ge f(0)x.$$

Therefore

$$\int_{-1}^{1} xf(x)dx = \int_{-1}^{1}\left(g(x) + \frac{x^2}{2}\right)dx \ge \int_{-1}^{1}\left(f(0)x + \frac{x^2}{2}\right)dx = \frac{1}{3}.$$

3. (IMC 2019 Problem 6) Let

$$h(x) = \int_0^x f(t)dt - g(x).$$

Since $f(x)$ is continuous, by fundamental theorem of calculus, we have $h'(x) = f(x) - g'(x)$ for any $x \in \mathbb{R}$. Using the assumption, we have

$$h'(0)h'(1) = (f(0) - g'(0))(f(1) - g'(1)) < 0.$$

So $h'(0)$ and $h'(1)$ are of opposite signs. By the mean value theorem for derivatives (Darboux's theorem), there exists $0 < c < 1$ such that $h'(c) = 0$ which means $f(c) = g'(c)$.

4. (IMC 2018 Problem 4) First we show that f is infinitely differentiable. For any $x \in (0, +\infty)$, by putting $a = \dfrac{x}{2}$ and $b = 2x$, we have

$$f'(x) = \frac{f(2x) - f(\frac{x}{2})}{\frac{3x}{2}}.$$

Since $f(x)$ is differentiable, we see that $f'(x)$ is differentiable which means $f(x)$ is twice differentiable. Using an inductive argument, we see that $f(x)$ is infinitely differentiable.

Now for any $t \in \mathbb{R}$, putting $b = e^t x$ and $a = e^{-t}x$, we have

$$f(e^t x) - f(e^{-t}x) = (e^t - e^{-t})xf'(x).$$

Differentiate the above equality with respect to t for 3 times, we have

$$\begin{aligned}
e^t xf'(e^t x) + e^{-t}xf'(e^{-t}x) &= (e^t + e^{-t})xf'(x) \\
e^{2t}x^2 f''(e^t x) + e^t xf'(e^t x) - e^{-2t}x^2 f''(e^{-t}x) - e^{-t}xf'(e^{-t}x) &= (e^t - e^{-t})xf'(x) \\
\begin{array}{r} e^{3t}x^3 f'''(e^t x) + 3e^{3t}x^2 f''(e^t x) + e^t xf'(e^t x) \\ +e^{-3t}x^3 f'''(e^{-t}x) + 3e^{-3t}x^2 f''(e^{-t}x) + e^{-t}xf'(e^{-t}x) \end{array} &= (e^t + e^{-t})xf'(x)
\end{aligned}$$

Now putting $t = 0$, we obtain

$$\begin{aligned}
\begin{array}{r} x^3 f'''(x) + 3x^2 f''(x) + xf'(x) \\ +x^3 f'''(x) + 3x^2 f''(x) + xf'(x) \end{array} &= 2xf'(x) \\
2x^3 f'''(x) + 6x^2 f''(x) &= 0 \\
xf'''(x) + 3f''(x) &= 0 \\
(xf(x))''' &= 0.
\end{aligned}$$

It follows that $xf(x) = C_2 x^2 + C_1 x + C_0$ and therefore

$$f(x) = C_2 x + C_1 + \frac{C_0}{x}$$

where C_0, C_1, C_2 are arbitrary constants. It is easy to verify that all functions of this form satisfy the condition.

5. (Putnam 2015 B1) Let $g(x) = e^{\frac{x}{2}} f(x)$. Then g has at least 5 distinct real zeros. By Rolle's theorem, $g'(x)$ has at least 4 distinct zeros. By repeating the argument, $g''(x)$ has at least 3 distinct zeros and $g'''(x)$ has at least 2 zeros. Now

$$\begin{aligned} g'(x) &= \frac{1}{2}e^{\frac{x}{2}}(f(x) + 2f'(x)) \\ g''(x) &= \frac{1}{4}e^{\frac{x}{2}}(f(x) + 4f'(x) + 4f''(x)) \\ g'''(x) &= \frac{1}{8}e^{\frac{x}{2}}(f(x) + 6f'(x) + 12f''(x) + 8f'''(x)). \end{aligned}$$

Thus $f(x) + 6f'(x) + 12f''(x) + 8f'''(x) = 8e^{-\frac{x}{2}}g'''(x)$ has at least 2 distinct zeros.

6. (Putnam 1997 B2) Let $h = f^2 + (f')^2$. Then

$$h' = 2ff' + 2f'f'' = 2f'(f + f'') = -2xg(f')^2.$$

Thus $h'(x) \geq 0$ when $x < 0$ and $h'(x) \leq 0$ when $x > 0$. It follows that $h(x) \leq h(0)$ for any $x \in \mathbb{R}$. Therefore

$$f^2 = h^2 - (f')^2 \leq h^2 \leq (h(0))^2$$

which implies $|f(x)|$ is bounded.

7. (VJMC 2019 II P2) Let $g(x) = \sin(f(x))$. Then

$$\begin{aligned} g' &= f' \cos f \\ g'' &= f'' \cos f - (f')^2 \sin f \geq 0. \end{aligned}$$

Suppose there exists $a \in \mathbb{R}$ such that $g'(a) \neq 0$. For any $x \in \mathbb{R}$, we can find ξ between a and x, by Taylor's theorem, which shows that

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(\xi)(x-a)^2}{2!} \leq g(a) + g'(a)(x-a).$$

Now by taking $x = a - \dfrac{2 + g(a)}{g'(a)}$, we have

$$g(x) \leq g(a) + g'(a)\left(a - \frac{2+g(a)}{g'(a)} - a\right) = -2$$

which is impossible since $g(x) = \sin(f(x)) \geq -1$ for any $x \in \mathbb{R}$. Therefore $g(x)$ is a constant function which implies $f(x)$ is a constant function. It is easy to see that any constant function satisfies the condition.

8. (VJMC 2013 I P1) Let $g = f^2 - 2\sin x$. Then

$$\begin{aligned} |g| &\leq f^2 + 2 \leq M^2 + 2 \\ g' &= 2ff' - 2\cos x \geq 0. \end{aligned}$$

So $g(x)$ is a bounded monotonic increasing function which implies $\lim\limits_{x\to+\infty} g(x)$ exists. It follows that $\lim\limits_{x\to+\infty} (f(x))^2 = \lim\limits_{x\to+\infty} (g(x) + \sin x)$ does not exist. Therefore $\lim\limits_{x\to+\infty} f(x)$ does not exist.