IMC 2023 Training
Combinatorics Practice Problems

1 Two-Player Games

Problem 1.1 (Putnam 2020)**.** Let k, n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any unoccupied hole to its right. Alice and Bob alternate moves with Alice going first. The game ends when the pegs occupy the k rightmost holes, and whoever moves next loses. For what values of k and n does Alice have a winning strategy?

Problem 1.2.* Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones, with Alice going first. Whoever takes the last stone wins. Prove that for each of the following two game rules, there are infinitely many positive integers n such that Bob has a winning strategy.

a) (Waterloo 2016) The number of stones removed in each turn must be a non-zero perfect square.

b) (Putnam 2006) The number of stones removed in each turn must be 1 less than a prime number.

Problem 1.3. A general Nim game starts with k piles of stones of size $a_1, \cdots, a_k$. Alice and Bob take turn to pick 1 pile and remove some non-zero amount of stones from it, with Alice going first. The number of stones that can be removed in each turn depends on the game rule. Whoever takes the last stone wins. These games can be solved by inductively assigning a nim-value to each possible game state as follows. The end state with no stones left is assigned nim-value 0. Any other game state is assigned the smallest non-negative integer not already assigned to a state that can be reached with a valid move.

a) Show that a game state is in W iff it is assigned a non-zero nim-value. Equivalently, a game state is in L iff it is assigned nim-value 0.

b) For a fixed general Nim game, if the nim-value of k piles of stones of sizes $a_1, \cdots, a_k$ is x and the nim-value of ℓ piles of stones of sizes $b_1, \cdots, b_\ell$ is y, then the nim-value of $k + \ell$ piles of stones of sizes $a_1, \cdots, a_k, b_1, \cdots, b_\ell$ is $x \oplus y$, where $\oplus$ is the binary XOR addition. This provides a way to easily compute nim-values. Either take it as fact, look up a proof online, or (hard) prove it yourself.

c) Determine who has a winning strategy in the Nim game with 4 piles of stones of size 1,3,5,7, and the rule that each turn a player can remove any non-zero number of stones from a pile.

d) (Putnam 1995) Determine who has a winning strategy in the Nim game with 4 piles of stones of size 3,4,5,6, and the game rule that each turn a player can either

- take 1 stone from a pile, provided that at least 2 stones remain afterwards, or
- take an entire pile of 2 or 3 stones.

Problem 1.4.* For positive integers m, n, k, an (m, n, k)-game involves two players taking turns in placing a stone of their colour on an $m \times n$ board. The first player managing to place k stones of their colour in a row vertically, horizontally or diagonally wins. If the entire board is covered without either player winning, the game ends in a tie. For example, the $(3, 3, 3)$-game is tic-tac-toe and the $(15, 15, 5)$-game is gomoku. Show via strategy stealing that in any (m, n, k)-game, the second player does not have a winning strategy. (It is not known for general (m, n, k) whether the first player has a winning strategy or both player can force a draw.)

2 Double Counting

Problem 2.1. Given integers $v, b, r, k, \lambda \geq 1$, a collection $\mathcal{S}$ of subsets of $\{1, 2, \cdots, v\}$ is called a (v, b, r, k, λ)-design if the following conditions are satisfied:

- $|\mathcal{S}| = b$. In other words, $\mathcal{S}$ consists of b subsets of $\{1, 2, \cdots, v\}$.
- Each set in $\mathcal{S}$ has size k.
- For each $j \in \{1, 2, \cdots, v\}$, there are exactly r sets in $\mathcal{S}$ containing j.
- For any distinct $j_1, j_2 \in \{1, 2, \cdots, v\}$, there are exactly λ sets in $\mathcal{S}$ containing both j_1 and j_2.

Show that if a (v, b, r, k, λ)-design exists, then we must have $bk = vr$ and $\lambda(v-1) = r(k-1)$. (It is known that a (43,43,7,7,1)-design does not exist, so these two conditions are not sufficient.)

Problem 2.2 (IMC 2022)**.** Colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue, so that every vertex is the end point of 20 red edges and 22 blue edges. A triangle formed with 3 vertices of P is monochromatic if all 3 of its edges have the same colour. If there are 2022 blue monochromatic triangles, how many red monochromatic triangles are there?

Problem 2.3*(Russia 1996)**.** 1600 delegates formed 16000 committees of 80 members each. Prove that there exists 2 committees with at least 4 common members.

Problem 2.4*(IMC 2009)**.** In a town with n residents, any 2 who are not friends have at least 1 common friend, and no one is a friend of everyone else. Number them from 1 to n and let a_i be the number of friends of resident i. Suppose $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least 3) that can be seated around a table so that any two neighbours are friends. Determine all possible values of k.

Problem 2.5. Let p and q be coprime positive integers. Show that $\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \sum_{j=1}^{p-1} \left\lfloor \frac{jq}{p} \right\rfloor$.

Problem 2.6*(IMO 1989)**.** Let n and k be positive integers. Let S be a set of n points in the plane such that no three points in S are colinear, and for every point $p \in S$, there are at least k other points in S whose distance to p are the same. Show that $k < \frac{1}{2} + \sqrt{2n}$.

3 Binomial Identities

Problem 3.1. Prove the following Binomial Identites. Try to think of multiple methods.

a) For integers $n \geq k \geq 0$, $\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$. Equivalently, for integers $n, m \geq 0$, $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

b) For integers $n \geq k \geq j \geq 0$, $\sum_{m=j}^{n-k+j} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$. Note that setting $k = j$ gives the first identity in a).

c) For integers $n > k \geq 0$, $\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$.

d) For integers $n \geq q \geq 0$, $\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$.

e) For integers $n \geq 0$, $\sum_{k=0}^{n} k^2 \binom{n}{k} = 2^{n-2}(n^2 + n)$ and $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(2^{n+1} - 1)$.

Problem 3.2*. For integers m, n satisfying $0 \leq m \leq 2n+1$, show that $\sum_{k=0}^{m} 2^k \binom{n}{k}\binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} = \binom{2n+1}{m}$.

Problem 3.3. Let $\omega = e^{\frac{2\pi i}{3}}$. By substituting $x = 1, \omega, \omega^2$ into $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, show that $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} = \frac{1}{3}(2^n + 2\cos\frac{n\pi}{3})$. Can you generalise this further?

Problem 3.4. Let F_n be the n-th Fibonacci number starting with $F_0 = F_1 = 1$. Show that $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_n$ for all $n \geq 0$.

Problem 3.5. For any $\alpha \in \mathbb{C}$ and non-negative integer k, the generalised binomial coefficient $\binom{\alpha}{k}$ is defined to be $\binom{\alpha}{k} = \frac{1}{k!}\alpha(\alpha-1)\cdots(\alpha-k+1)$. With this definition, it is known that the Generalised Binomial Theorem $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ holds for every $x \in \mathbb{C}$ with $|x| < 1$.

a) Show that when $\alpha \in \mathbb{Z}_{\geq 0}$, this generalised definition and biniomial theorem agrees with the usual ones.

b) Show that $(-4)^k \binom{-\frac{1}{2}}{k} = \binom{2k}{k}$ for every non-negative integer k. Hence $(1-4x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$.

c) Show that $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$ for every non-negative integer n.

d) Show that $(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{k+r-1}{k} x^k$ for every non-negative integer r. This is related to the negative binomial distribution $\mathrm{NB}(r, p)$.

Problem 3.6 (IMC 2020)**.** Let n be a positive integer. Compute the number of length n words w satisfying the following properties:

- All letters in w are from the alphabet $\{a, b, c, d\}$.
- w contains an even number of letter a, and an even number of letter b.

Problem 3.7*(IMC 2021)**.** Let $n, k \geq 1$ and $a \geq 0$ be integers. Choose a size k subset X of $\{1, 2, \cdots, k+a\}$ uniformly at random. Independently, choose a size n subset Y of $\{1, 2, \cdots, n+k+a\}$ uniformly at random. Show that the probability $\mathbb{P}(\min(Y) > \max(X))$ is independent of a.

Problem 3.8 (Putnam 2020)**.** Let k be a non-negative integer. Evaluate $\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}$.

Problem 3.9*(IMC 2002)**.** Let n be a positive integer. For $1 \leq k \leq n$, let $a_k = \frac{1}{\binom{n}{k}}$ and $b_k = 2^{k-n}$. Show that $\sum_{k=1}^{n} \frac{1}{k}(a_k - b_k) = 0$.

Problem 3.10*(Putnam 1987)**.** Let r, s, t be non-negative integers with $r + s \leq t$. Prove that $\sum_{j=0}^{s} \frac{\binom{s}{j}\binom{t-s}{r}}{\binom{t}{r+j}} = \frac{t+1}{t+1-s}$.

Problem 3.11*(IMC 2002)**.** For each $n \geq 1$, let $a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}$ and $b_n = \sum_{k=0}^{\infty}(-1)^k \frac{k^n}{k!}$. Show that $a_n b_n$ is always an integer.

Problem 3.12. The Inclusion-Exclusion Principle states that for finite sets $A_1, \cdots, A_n$, we have $|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq S \subset \{1,\cdots,n\}} (-1)^{|S|-1} \left|\bigcap_{j\in S} A_j\right|$.

a) A permutation σ on $\{1, 2, \cdots, n\}$ is a derangement if $\sigma(i) \neq i$ for all $1 \leq i \leq n$. Show that the number D_n of derangement on $\{1, 2, \cdots, n\}$ is $\sum_{k=0}^{n}(-1)^k \frac{n!}{k!}$.

b) Show that the number of surjective functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$ is given by $\sum_{k=0}^{n} \binom{n}{k}(-1)^k(n-k)^m$. In particular, this is equal to 0 if $n > m$, and equal to $n!$ if $n = m$.

Problem 3.13$^*_*$(IMC 2015)**.** Consider all 26^{26} words of length 26 in the Latin alphabet. Define the weight of a word as $\frac{1}{k+1}$, where k is the number of letters not used in this word. Prove that the weighted sum over all words is 3^{75}.

Problem 3.14$^*_*$(IMC 2014). Let n be a positive integer. Let D_n be the number of derangements on $\{1,\cdots,n\}$. For $1 \le k \le \frac{n}{2}$, let $\Delta(n,k)$ be the number of derangements σ on $\{1,\cdots,n\}$ with the additional conditions that $\sigma(i)=k+i$ for every $1\le i\le k$. Show that $\Delta(n,k)=\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{n+1-k-i}}{n-k-i}$.

Problem 3.15$^*_*$(Putnam 2005). Let S_n denote the set of all permutations on $\{1,\cdots,n\}$. For $\pi\in S_n$, let $\sigma(\pi)$ denote the sign of π and let $\nu(\pi)$ denote the number of fixed points of π. Show that $\sum_{\pi\in S_n}\frac{\sigma(\pi)}{\nu(\pi)+1}=(-1)^{n+1}\frac{n}{n+1}$.

4 Generating Functions

Problem 4.1 (2001 Putnam)**.** You have n coins $C_1,\cdots,C_n$. For each $1\le k\le n$, when tossing the coin C_k, it has probability $\frac{1}{2k+1}$ of showing head and probability $\frac{2k}{2k+1}$ of showing tail. What is the probability that when all n coins are tossed, the total number of heads is odd.

Problem 4.2. Let n,k be positive integers. A partition of n with k parts is a k-tuple $(\lambda_1,\cdots,\lambda_k)$ of integers such that $\lambda_1\ge\cdots\ge\lambda_k\ge 1$ and $\sum_{j=1}^{k}\lambda_j=n$. For example, there are a total of 5 partitions of 4, which are $(4),(3,1),(2,2),(2,1,1)$ and $(1,1,1,1)$.

a) Let $a_0=1$. For $n\ge 1$, let a_n be the number of partitions of n. Show that $\sum_{n=0}^{\infty}a_nx^n=\prod_{i=1}^{\infty}\frac{1}{1-x^i}$.

b) Let $b_0=1$. For $n\ge 1$, let b_n be the number of partitions of n such that each part is distinct. Show that $\sum_{n=0}^{\infty}b_nx^n=\prod_{i=1}^{\infty}(1+x^i)$.

c) Let $c_0=1$. For $n\ge 1$, let c_n be the number of partitions of n such that each part is odd. Show that $\sum_{n=0}^{\infty}c_nx^n=\prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$.

d) Show that $b_n=c_n$ for all $n\ge 1$.

e) Show that every non-negative integer n has a unique base-2 representation.

f) Show that every integer can be uniquely written as a sum of the form $\sum_{k=0}^{\infty}d_k3^k$, where $d_k\in\{-1,0,1\}$ for each k.

Problem 4.3*(HMMT 2007). Let S denote the set of all triples of positive integers (i,j,k) with $i+j+k=17$. Compute $\sum_{(i,j,k)\in S}ijk$.

Problem 4.4. Suppose $a_0=0$, $a_1=1$ and $a_{n+1}=5a_n-6a_{n-1}$ for all $n\ge 1$. Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$.

a) Show that $(1-5x+6x^2)f(x)=x$, and thus $f(x)=\frac{1}{1-3x}-\frac{1}{1-2x}$. Conclude that $a_n=3^n-2^n$ for all $n\ge 0$.

b) Use the same method to show that for any order 2 linear recurrence of the form $a_{n+1}=pa_n+qa_{n-1}$, if $x^2-px-q=0$ has distinct roots $\alpha,\beta\in\mathbb{C}$, then $a_n=A\alpha^n+B\beta^n$ for some A,B depending on a_0,a_1. In particular, find an explicit formula for the Fibonacci numbers F_n.

c) What happens if the two roots of $x^2-px-q=0$ coincide?

d) Try to generalise this to higher order linear recurrences.

Problem 4.5. The Catalan numbers C_n are given by $C_0=1$ and the recurrence $C_{n+1}=\sum_{i=0}^{n}C_iC_{n-i}$ for all $n\ge 0$. Let $f(x)=\sum_{n=0}^{\infty}C_nx^n$.

a) Show that $xf(x)^2-f(x)+1=0$, so $f(x)=\frac{1\pm\sqrt{1-4x}}{2x}$.

b) Show that $f(x)=\frac{1-\sqrt{1-4x}}{2x}$ by using $\lim_{x\to 0}f(x)=C_0=1$.

c) By doing a calculation similar to the one in Problem 3.5 for $(1-4x)^{\frac{1}{2}}$, show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Problem 4.6*(IMC 2003). Let $a_0 = 1$ and $a_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \frac{a_k}{n-k+2}$ for all $n \geq 0$. Find the limit $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k}$.

Problem 4.7$^*_*$(IMC 2019). Suppose $a_0 = 1$, $a_1 = 2$ and $(n+3)a_{n+2} = (6n+9)a_{n+1} - na_n$ for all $n \geq 0$. Show that all terms of this sequence are integers. (In fact, a_n are the large Schröder numbers.)

Problem 4.8*(Putnam 2005). For positive integers m and n, let $a_{m,n}$ denote the number of n-tuples $(x_1, \cdots, x_n)$ of integers such that $\sum_{i=1}^{n} |x_i| \leq m$. Show that $a_{m,n} = a_{n,m}$.

5 Hints for Selected Problems

Problem 1.2. Show that if N is the largest integer for which Bob has a winning strategy, and S is the set of number of stones allowed to be removed in each step, then for every $n > N$, one of $n-1, n-2, \cdots, n-N$ must be in S.
Problem 1.4. In (m,n,k)-games, it never hurts to have an extra stone of your colour on the board.
Problem 2.3. If $x_1, \cdots, x_n$ are non-negative integers such that $x_1 + \cdots + x_n = m$, what is the minimum possible value of $x_1^2 + \cdots + x_n^2$?
Problem 2.5. Double count the number of points with integer coordinates that are both inside and below the diagonal of the rectangle with vertices $(0,0), (p,0), (0,q), (p,q)$.
Problem 2.6. Double count the number of isosceles triangles with vertices in S.
Problem 3.2. View a set of $2n+1$ objects as n groups of 2 objects and 1 group of 1 object. Count the number of ways to pick m objects according to how many is chosen from each group. Alternatively, show that $\binom{n}{\lfloor \frac{m}{2} \rfloor}$ is the coefficient of x^{m-n} in $(x+\frac{1}{x})^n(1+x)$.
Problem 3.4. Induction.
Problem 3.5. For part c), square the power series for $(1-4x)^{-\frac{1}{2}}$ proven in b).
Problem 3.7. For integers i, j, count the number of X with $\max(X) = i$ and the number of Y with $\min(Y) = j$. Use this to express the probability as a double sum and simplify using various binomial identities. The official solution uses a different and easier method.
Problem 3.8. Guess the answer first, then prove it by induction.
Problem 3.9. Induction.
Problem 3.10. For each term in the sum, write out the binomial coefficients in factorial form. Then regroup these factorials into different binomial coefficients.
Problem 3.11. Try to find recurrence relations for (a_n) and (b_n).
Problem 3.13. Use Inclusion-Exclusion to count the number of words that uses exactly k letters.
Problem 3.14. Use Inclusion-Exclusion to find an exact formula for $\Delta(n,k)$. Then show both sides of the equation satisfy a same recurrence relation.
Problem 3.15. For $\pi \in S_n$, let $\text{Fix}(\pi)$ be the set of elements fixed by π. For $T \subset \{1, \cdots, n\}$, use Inclusion-Exclusion to express $\sum_{\text{Fix}(\pi)=T} \sigma(\pi)$ in terms of sums of the form $\sum_{\text{Fix}(\pi)\supset R} \sigma(\pi)$ with $R \supset T$. Then show $\sum_{\text{Fix}(\pi)\supset R} \sigma(\pi)$ is often 0.
Problem 4.3. Try to use $f(x) = \sum_{n=0}^{\infty} nx^n$.
Problem 4.6. Show that the generating function $f(x)$ satisfies the differential equation $f'(x) = f(x)\sum_{k=0}^{\infty} \frac{x^k}{k+2}$. Then solve for $f(x)$.
Problem 4.7. The generating function $f(x)$ satisfies a differential equation. Use it to solve for $f(x)$. Then show that $f(x)$ satisfies another functional equation and use this to give an alternate recurrence formula for a_n.
Problem 4.8. Consider the two-variable generating function $f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}x^m y^n$.