IMC 2023 training session
Combinatorics

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Outline

Example (The Nim Game)

There are two piles of stones of size $m, n \in \mathbb{Z}^+$. Alice and Bob alternates with Alice moving first. Each turn, they can remove any non-zero number of stones, provided that the stones are all from the same pile. The one who removes the last stone wins. Who has a winning strategy?

Example (The Game of Chomp)

Alice and Bob play a game on an $m \times n$ board. They alternate with Alice going first. Each turn, they pick a square on the board and remove it along with all the squares to the right and bottom of it. Whoever takes the top-left square loses. Show that Alice has a winning strategy unless $(m, n) = (1, 1)$.

Two-Player Games

Theorem (Zermelo's Theorem)

If a two-player game satisfy the following conditions:

- *The players alternate moves.*
- *The game has perfect information. In other words, both players know all previous moves and all possible future moves.*
- *The game has finitely many possible game states and cannot end in a tie.*
- *There are no randomness involved.*

Then one of the players must have a winning strategy.

Remark

If the game can end in a tie, then either one of the player has a winning strategy, or both player can force a tie.

Two-Player Games

Definition

Let W be the set of games states such that the player moving next wins, and let L be those the player moving next loses.

Remark

A consequence of Zermelo's Theorem is that we can partition all the possible game states into these two disjoint sets W and L.
Note that a game state is in W iff there is a legal move to a game state in L, and a game state is in L if and only if every legal move result in a game state in W.
Common approaches to decide which player has a winning strategy:

- *Working backwards from the simplest cases to decide whether each game state is in W or L, until we reach the initial game state.*
- *The strategy stealing argument.*

Analyse W and L

Example (The Nim Game)

There are two piles of stones of size $m, n \in \mathbb{Z}^+$. Alice and Bob alternates with Alice moving first. Each turn, they can remove any non-zero number of stones, provided that the stones are all from the same pile. The one who removes the last stone wins. Who has a winning strategy?

Solution

It is clear that $(0,0),(1,1) \in L$ *and* $(n,0),(0,n) \in W$ *for all* $n \geq 1$.
Hence $(1,n),(n,1) \in W$ *for all* $n \geq 2$ *as the player facing this state can move so that the other player faces* $(1,1) \in L$.
Then it follows $(2,2) \in L$ *as any legal move leaves the other player with one of* $(2,0),(2,1),(1,2),(0,2)$, *which are all in* W. *Then* $(2,n),(n,2) \in W$ *for all* $n \geq 3$.
Induction shows $(m,n) \in W$ *iff* $m \neq n$. *So Alice wins iff* $m \neq n$.

Strategy Stealing

Example (The Game of Chomp)

Alice and Bob play a game on an $m \times n$ board. They alternate with Alice going first. Each turn, they pick a square on the board and remove it along with all the squares to the right and bottom of it. Whoever takes the top-left square loses. Show that Alice has a winning strategy unless $(m, n) = (1, 1)$.

Proof.

Suppose Bob has a winning strategy. In other words, the full $m \times n$ board is in L. Then any possible first move by Alice ends in W. In particular, the $m \times n$ board without the bottom-right square is in W. This means Bob has a move to turn this into something in L. However, Alice can just steal this strategy and make this move in the beginning instead and leave Bob with this state in L, contradiction. □

Example (IMC 2002)

200 students participate in a maths contest which consists of 6 questions. If each question is solved by at least 120 students, show that there exists two students such that each question is solved by at least one of them.

Proof.

Let S be the set of students, Q be the set of questions and let N be the number of triples $(i, j, k) \in S \times S \times Q$ such that $i \neq j$ and students i, j both fail to solve question k.

Suppose for a contradiction that for any two students $i \neq j$, there is a question k both of them can't solve, then this implies $N \geq 200 \times 199 = 39800$. On the other hand, for each question k, at most 80 students didn't solve it, so $N \leq 80 \times 79 \times 6 = 37920$, contradiction. □

Example

1500 people attend an event. Suppose each of them is friends with at least 100 and at most 201 other people present. Show that we can find 500 of them which can be split into 250 pairs of friends.

Proof.

Let M be a maximal set of people that can be split into $\frac{|M|}{2}$ pairs of friends. Let L be the set of people remaining. Let N be the number of pairs of friends $(i, j) \in M \times L$.

On the one hand, if any $j \in L$ is friends with another $j' \in L$, then we can add j, j' to M to get a larger set satisfying the required condition, contradiction. So for every $j \in L$, all of his/her friends are in M. Hence $N \geq 100|L|$. On the other hand, every $i \in M$ can be friends with at most 200 people in L, so $N \leq 200|M|$. Hence $|L| \leq 2|M|$ and $|M| \geq 500$. □

Binomial Coefficients

Definition (Binomial Coefficients)

For integers $n, k \geq 0$, the binomial coefficient $\binom{n}{k}$ can be defined in one of the following equivalent ways:

- Combinatorics: $\binom{n}{k}$ is the number of ways to pick a set of k elements from a set of n elements.
- Algebra: if $n \geq k$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- Binomial Theorem: $\binom{n}{k}$ is the coefficient of the x^k term in $(1+x)^n$. In other words, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Example

- $\binom{n}{0} = \binom{n}{n} = 1$.
- $\binom{n}{k} = \binom{n}{n-k}$ if $n \geq k \geq 0$.
- $\binom{n}{k} = 0$ if $k > n$.

Proving Binomial Identities Combinatorially

Example

For non-negative integers n, m, k with $n \geq m$ and $n \geq k$, we have $\binom{n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j}$.

Proof.

Method 1: Suppose we are trying to select a k person committee from a group of m men and $n - m$ women. Ignoring gender, there are $\binom{n}{k}$ ways to do this. If there are j men and $k - j$ women in this committee, then there are $\binom{m}{j}$ ways to pick the men and $\binom{n-m}{k-j}$ ways to pick the women.

Method 2: Compare the coefficient of x^k on both sides of the equation $(1+x)^n = (1+x)^m(1+x)^{n-m}$. □

Proving Binomial Identities Algebraically

The common techniques for proving binomial identities algebraically are:

- Manipulate $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ algebraically (differentiate, integrate, substitute).
- Induction (after guessing the result).
- Use known identities. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ and $k\binom{n}{k} = n\binom{n-1}{k-1}$ are especially helpful for induction.
- Exchange the order of summation.
- Regroup the factorials involved in the binomial coefficients.

Proving Binomial Identities Algebraically

Example

For $n \geq 1$, substituting $x = 1$ and $x = -1$ into $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, we get $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Let $E = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}$ be the sum of the even binomial coefficients, and $O = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1}$ be the sum of the odd binomial coefficients. Then the two equations above says $E + O = 2^n$ and $E - O = 0$. Hence $E = O = 2^{n-1}$.

Proving Binomial Identities Algebraically

Example

Show that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \geq 0$.

Proof.

Method 1:
$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}$.
Method 2:
Differentiate $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, we get
$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}$. Substitute $x = 1$.
Method 3: (Combinatorially)
Suppose we are picking a team of any size from a group of n people, and 1 captain from the team. Left hand side counts this by picking the team first, then the captain. Right hand side counts this by picking the captain first, then the remaining team. □

Generating Functions

Definition

Given a sequence (a_n), the generating function for (a_n) is $f(x) = \sum_n a_n x^n$.

Example

- The generating function for the sequence $(a_0, a_1, a_2, \cdots) = (1, 1, 1, \cdots)$ is $\sum_{n \geq 0} x^n = \frac{1}{1-x}$.
- Differentiate the above equation on both sides, we get $\sum_{n \geq 1} nx^{n-1} = \frac{1}{(1-x)^2}$. So the generating function for the sequence $(a_0, a_1, a_2, \cdots) = (0, 1, 2, \cdots)$ is $\sum_{n \geq 0} nx^n = \frac{x}{(1-x)^2}$.
- The generating function for the sequence $(\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n})$ is $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$.

Example (IMC 2022)

Let p be a prime number. A flea starts at point 0 on $\mathbb{R}$. Every minute, the flea can either move by 1 to the left or right, or stay put. Let $N(p)$ be the number of ways for the flea to return to 0 after $p-1$ minutes. Find $N(p)$ modulo p.

Solution

Fix a prime p. For $-(p-1) \leq n \leq p-1$*, let* a_n *be the number of ways for the flea to end up at point n after* $p-1$ *minutes. Then* $f(x) = \sum_{n=-(p-1)}^{p-1} a_n x^n = (x^{-1}+1+x)^{p-1} = x^{1-p}(1+x+x^2)^{p-1}$*. So* $N(p) = a_0$ *is the coefficient of* x^{p-1} *in* $(1+x+x^2)^{p-1}$*.*
The rest is algebra, and is omitted here.

Example

Let $a_0 = 0$ and $a_n = 2a_{n-1} + n$ for all $n \geq 1$. Find a formula for a_n.

Solution

The generating function for (a_n) *is* $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
We have
$f(x) = \sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} nx^n = 2xf(x) + \frac{x}{(1-x)^2}$.

So $f(x) = \frac{x}{(1-x)^2(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} =$
$2\sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(n+1)x^n$.

Hence $a_n = 2^{n+1} - n - 2$.