Some Practice Problems on Polynomials

Problem 1 Represent

$$S(x_1, \dots, x_n) = \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} x_i^3 x_j$$

as a polynomial in the elementary symmetric polynomials.

Problem 2 Let $P(x) = x^3 + ax^2 + bx + c$ have roots x_1, x_2, x_3. Find the polynomial $Q(x) = x^3 + \alpha x^2 + \beta x + \gamma$ of degree 3 that has roots x_1x_2, x_1x_3, x_2x_3.

Problem 3 Let $n \geq 3$ and $x_1, \dots, x_n \in \mathbb{C}$ be all roots of the polynomial

$$p(x) = x^n - 2x^{n-1} + 3x - 1.$$

What is

$$\sum_{i \neq j} \frac{x_i}{x_j},$$

where the sum is taken over all ordered pairs $(i, j) \in \{1, \dots, n\}$ except for $i = j$.

Problem 4 Find all pairs of integers $m, n \in \mathbb{N}$ such that the polynomial

$$P(x) = 1 + x + x^2 + \dots + x^m$$

divides the polynomial

$$Q(x) = 1 + x^n + x^{2n} \dots + x^{mn}.$$