IMC Selection Test 2

Solution to Problem 1. Answer: 2^{n-1}. This is clearly true for $n = 1$ and we can use induction on n to prove this. Let $n \geq 2$. For any sequence $(a_1, \ldots, a_n)$ as in the question, we remove the first element a_1 and replace each entry $a_j > a_i$ by $a_j - 1$. This way we obtain a legal sequence for $n-1$. On the other hand, any $(n-1)$-sequence is obtained precisely 2 times: if it starts with $a \in \{1, \ldots, n-1\}$ we can "reinsert" either a or $a+1$. Thus the number of legal sequences for n is twice as large as for $n-1$.

An alternative solution can be obtained by proving (e.g. via induction on n) that each legal sequence ends in 1 or n.

Solution to Problem 2. If a_0 is a fixed point for f, we are done. Thus we may assume $f(a_0) \neq a_0$, and can furthermore assume wlog $f(a_0) > a_0$.

Now if $f(x) \leq x$ for some $x > a_0$, then the Intermediate Value Theorem applied to $g(x) := f(x) - x$ implies f has a fixed point at some $c \in [a_0, x]$. Thus, we may assume $f(x) > x$ for all $x > a_0$, and so clearly $a_{n+1} > a_n$ for all n.

We now prove that the sequence (a_n) is bounded by $L := \lim_{n\to\infty} \frac{a_0 + \cdots + a_n}{n}$ and thus converges. Assume for contradiction that $a_\ell > L$ for some $\ell \geq 0$. Therefore $a_i > a_\ell$ for all $i > \ell$, and thus

$$\frac{a_0 + \cdots + a_i}{i} = \frac{a_0 + \cdots + a_\ell}{i} + \frac{a_{\ell+1} + \cdots + a_i}{i} > \frac{a_0 + \cdots + a_\ell}{i} + \frac{(i - \ell + 1)a_\ell}{i}$$

for any $i \geq \ell$. Taking the limit as $i \to \infty$ yields $L \geq a_\ell$, which is a contradiction. Thus (a_n) is an increasing bounded sequence, and therefore converges to some limit $a \in \mathbb{R}$. Thus, as f is continuous, a is a fixed point of f.

Solution to Problem 3.

$$\frac{(r-1)(4a_r - ra_{r-1})a_{r-2}}{a_r a_{r-1}} = 4 \Rightarrow$$

$$\frac{4(r-1)a_r - (r-1)ra_{r-1}}{a_r a_{r-1}} = \frac{4}{a_{r-2}} \Rightarrow$$

$$\frac{4(r-1)}{a_{r-1}} - \frac{(r-1)r}{a_r} = \frac{4}{a_{r-2}} \Rightarrow \frac{4(r-1)!}{a_{r-1}} - \frac{r!}{a_r} = \frac{4(r-2)!}{a_{r-2}}.$$

By setting $b_r := \frac{r!}{a_r}$ and solving the characteristic equation we find that $b_r = c_1 2^r + c_2 r 2^r$, and by the initial conditions we deduce that $b_r = 2^r \Rightarrow a_r = \frac{r!}{2^r}$. By Legendre's formula, the exponent of 2 in $r!$ is $\sum_{i=1}^{\infty} \lfloor \frac{r}{2^i} \rfloor < r$, therefore a_r is never an integer for $r > 0$.

Solution to Problem 4. We have

$$S = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n!}{(r!)^2(n-2r)!} 2^{n-2r} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{r}\binom{n-r}{r} 2^{n-2r}.$$

Thus S is the constant term (that is, the coefficient at x^0) when we expand $(x+1/x+2)^n$. Indeed, in order to get x^0, we have for some r with $0 \le r \le n/2$ pick r times x, having $\binom{n}{r}$ choices here, then pick r times $1/x$ (having $\binom{n-r}{r}$ choices) with all these choices giving the same contribution 2^{n-2r} to the constant term.

Since $(x + 1/x + 2)^n = (1 + x)^{2n}/x^n$, S is the coefficient at x^n in $(1 + x)^{2n}$ which is $\binom{2n}{n}$ by the Binomial Theorem.