IMC Selection Test 3

Solution to Problem 1. For any $n \geq 1$, we can define positive integers A_n, B_n such that $(1+\sqrt{7})^n = A_n + B_n\sqrt{7}$. We prove by induction that $\gcd(A_{2k+1}, B_{2k+1}) = 2^k$ for all $k \geq 0$. Clearly $\gcd(A_1, B_1) = 1$. Now assume $\gcd(A_{2k+1}, B_{2k+1}) = 2^k$ for some $k \geq 0$. We note that

$$\begin{aligned} A_{2k+3} + B_{2k+3}\sqrt{7} &= (8+2\sqrt{7})(A_{2k+1} + B_{2k+1}\sqrt{7}) \\ &= (8A_{2k+1} + 14B_{2k+1}) + (2A_{2k+1} + 8B_{2k+1})\sqrt{7} \end{aligned}$$

thus $A_{2k+3} = 8A_{2k+1} + 14B_{2k+1}$ and $B_{2k+3} = 2A_{2k+1} + 8B_{2k+1}$.

Firstly, its clear by induction that $A_{2k+1}, B_{2k+1} \equiv 1 \pmod 3$, for all $k \geq 0$, thus 3 cannot divide $\gcd(A_{2k+1}, B_{2k+1})$ for any k. Now, for any positive integer d, we have

$$\begin{aligned} d \,|\, \gcd(A_{2k+3}, B_{2k+3}) &\iff d \,|\, 8A_{2k+1} + 14B_{2k+1} \quad \text{and} \quad d \,|\, 2A_{2k+1} + 8B_{2k+1} \\ &\iff d \,|\, 2A_{2k+1} \quad \text{and} \quad d \,|\, 2B_{2k+1} \qquad (\text{as } 3 \nmid d) \\ &\iff d \,|\, 2\gcd(A_{2k+1}, B_{2k+1}) \end{aligned}$$

which proves $\gcd(A_{2k+3}, B_{2k+3}) = 2\gcd(A_{2k+1}, B_{2k+1}) = 2^{k+1}$, thus proving the induction step. Therefore, $\gcd(A_{2023}, B_{2023}) = 2^{1011}$.

Solution to Problem 2. The matrix

$$M = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & A & \\ -1 & & & \end{pmatrix}$$

is a skew-symmetric matrix of odd order and, therefore, its determinant vanishes (which follows from $\det M = \det M^T = \det(-M) = -\det M$). Let J denote the all-1 matrix. Thus we have for every $x \in \mathbb{R}$ that

$$\begin{aligned} \det A &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -x & & & \\ \vdots & & A & \\ -x & & & \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -x & & & \\ \vdots & & A & \\ -x & & & \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -x & & & \\ \vdots & & A & \\ -x & & & \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -x & & & \\ \vdots & & A & \\ -x & & & \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -x & & & \\ \vdots & & A+xJ & \\ -x & & & \end{pmatrix} = \det(A+xJ), \end{aligned}$$

as required.

Here is another solution. Let $f(x) = \det(A + xJ)$. Then f is a polynomial. On the one hand, since n is even, we have $f(-x) = \det(A - xJ) = \det(A^T - xJ^T) = \det(-A - xJ) =$

$\det(A + xJ) = f(x)$. Thus $f(x)$ has only even degree terms. On the other hand, by subtracting the first row from all other rows, we also have

$$f(x) = \det(A + xJ) = \det \begin{pmatrix} a_{11} + x & a_{12} + x & \cdots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \cdots & a_{2n} + x \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} + x & a_{n2} + x & \cdots & a_{nn} + x \end{pmatrix}$$

$$= \det \begin{pmatrix} a_{11} + x & a_{12} + x & \cdots & a_{1n} + x \\ a_{21} - a_{11} & a_{22} - a_{12} & \cdots & a_{2n} - a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} - a_{11} & a_{n2} - a_{12} & \cdots & a_{nn} - a_{1n} \end{pmatrix}.$$

So $f(x)$ is a polynomial of degree at most 1. Hence $f(x) = \det(A + xJ)$ must be a constant.

Solution to Problem 3. We show that the image of g is $(\log 2, \infty)$.

First, we show g is strictly increasing. Let $F : (0, \infty) \to \mathbb{R}$ be given by $F(x) = \int_1^x \frac{f(t)}{t} \mathrm{d}t$. Then by the Fundamental Theorem of Calculus, $F(x)$ is differentiable with $F'(x) = \frac{f(x)}{x}$. Since $g(x) = F(2x) - F(x)$, we have g is differentiable and $g'(x) = 2F'(2x) - F'(x) = \frac{f(2x) - f(x)}{x} > 0$, as required. Therefore, it suffices to show that $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to 0^+} g(x) = \log 2$.

We have $g(x) = \int_x^{2x} \frac{f(t)}{t} \mathrm{d}t > \int_x^{2x} \frac{f(x)}{2x} \mathrm{d}t = \frac{f(x)}{2}$. So $\lim_{x\to\infty} g(x) \geq \frac{1}{2} \lim_{x\to\infty} f(x) = \infty$. A more careful estimate gives $g(x) = \int_x^{2x} \frac{f(t)}{t} \mathrm{d}t < f(2x) \int_x^{2x} \frac{1}{t} \mathrm{d}t = f(2x) \log(\frac{2x}{x}) = f(2x) \log 2$ and $g(x) = \int_x^{2x} \frac{f(t)}{t} \mathrm{d}t > f(x) \int_x^{2x} \frac{1}{t} \mathrm{d}t = f(x) \log(\frac{2x}{x}) = f(x) \log 2$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(2x) = 1$, by Squeeze Theorem we have $\lim_{x\to 0^+} g(x) = \log 2$.

Solution to Problem 4. Take $n > 2023$.

By Ramsey's Theorem, there exists some number $R_1(n)$ such that every set of $R_1(n)$ regions contains a subset of n regions which pairwise intersect (or are pairwise disjoint, but by (iii) we know this not to occur).

By Ramsey's theorem for hypergraphs, there exists some number $R_2(n)$ such that every set of $R_2(n)$ regions contains a subset of n regions such that the intersection of any three of them is empty (or the intersection of any three of them is non-empty, but by Helly's theorem and (iv) we know this not to occur).

Let $R(n) := \max\{R_1(n), R_2(n)\}$ and consider a set of $R(n)$ regions of the family. Then this contains n regions that pairwise intersect yet no three of which intersect. These cover in total an area of size $nP_1 - \binom{n}{2} P_2$, which is negative for large enough n, unless $P_2 = 0$, which must therefore be true.

Suppose that one of the regions contains three non-collinear points. Then $P_1 > 0$, so the same holds for every region. Since $P_2 = 0$, this implies that the pairwise intersections are at the boundaries of the n regions. Consider the graph that has the n regions as its vertices and has an edge between every two tangent regions. Then this graph must be planar, a contradiction, since it is K_n.

We conclude that none of the regions contains three non-collinear points, hence each "region" is in fact a line segment. Indeed, a family like this can be made, for example, by considering long enough segments on the set of lines $y = a_k x + k$ with $k \geq 0$, $a_1 > 0$ and $a_k := a_{k-1} + p_k$, where p_k is the k^{th} prime number.