Some Practice Problems on Polynomials

Problem 1 Solution: The only degree sequence (ordered and with zeros ignored) in S is $(3,1)$. The corresponding diagram has one row of length 2 and two rows of length 1. Thus, as the first approximation we take $\sigma_1^2\sigma_2$. We have

$$S(\mathbf{x}) - \sigma_1^2(\mathbf{x})\sigma_2(\mathbf{x}) = -2\sum_{i<j} x_i^2 x_j^2 - 5\sum_{j<h;i\neq j;i\neq h} x_i^2 x_j x_h - 12\sigma_4(\mathbf{x}).$$

Here, the lex-highest degree sequence is $(2,2)$ and it results in the square 2×2 diagram. Thus we take σ_2^2. Using

$$\sum_{i<j} x_i^2 x_j^2 - \sigma_2^2(\mathbf{x}) = -2\sum_{j<h;i\neq j;i\neq h} x_i^2 x_j x_h - 6\sigma_4(\mathbf{x}),$$

we obtain that

$$S(\mathbf{x}) - \sigma_1^2(\mathbf{x})\sigma_2(\mathbf{x}) + 2\sigma_2^2(\mathbf{x}) = -\sum_{j<h;i\neq j;i\neq h} x_i^2 x_j x_h.$$

Here $(2,1,1)$ is the unique degree sequence. The columns of heights $2+1+1$ give rows of widths $3+1$, so we take $\sigma_1\sigma_3$. We have

$$\sum_{j<h;i\neq j;i\neq h} x_i^2 x_j x_h - \sigma_1(\mathbf{x})\sigma_3(\mathbf{x}) = -4\sigma_4(\mathbf{x}).$$

So, the answer is

$$S = \sigma_1^2\sigma_2 + 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4.$$

▮

Problem 2 Solution: We have $\alpha = -(x_1x_2 + x_1x_3 + x_2x_3) = -b$, $\gamma = -(x_1x_2x_3)^2 = -c^2$, and

$$\beta = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 = \sigma_3\sigma_1 = ac.$$

Thus, $Q(x) = x^3 - bx^2 + acx - c^2$. ▮

Problem 3 Solution: For $i \in \{1,\ldots,n\}$, let $\sigma_i(\mathbf{x}) = \sum_{I\subseteq[n]:|I|=i}\prod_{j\in I} x_j$ be the *i-th symmetric polynomial*; it is equal to $(-1)^i$ times the coefficient in front of x^{n-i} in p. We have, for example, $\sigma_1(\mathbf{x}) = x_1 + \ldots + x_n = 2$, $\sigma_{n-1}(\mathbf{x}) = (-1)^{n-1}\cdot 3$ and $\sigma_n(\mathbf{x}) = x_1\ldots x_n = (-1)^n(-1) = (-1)^{n+1}$.

Observe that, since $\sigma_n(\mathbf{x}) \neq 0$, no x_i is zero. Thus the sum in question is well-defined.

Thus the sum is equal to

$$\begin{aligned}\sum_{i\neq j}\frac{x_i}{x_j} &= -n + \sum_{i,j}\frac{x_i}{x_j} = -n + \sum_{j=1}^{n}\frac{x_1+\ldots+x_n}{x_j}\\ &= -n + \sigma_1(\mathbf{x})\sum_{j=1}^{n}\frac{1}{x_j} = -n + \sigma_1(\mathbf{x})\frac{\sigma_{n-1}(\mathbf{x})}{\sigma_n(\mathbf{x})} = -n + 6.\end{aligned}$$

▮

Problem 4 Solution: The polynomials P and Q do not have multiple roots since $(x-1)P(x) = x^{m+1} - 1$ and $(x^n - 1)Q(x) = x^{n(m+1)} - 1$ do not. How to show the last claim? One way is to observe that $x_l = \mathrm{e}^{2\pi i l/k}$, $l = 0, \ldots, k-1$, are k distinct roots of $R(x) = x^k - 1$. Since the degree of R is k, there are no other roots (and no multiple roots). Alternatively, R and its derivative kx^{k-1} have no common root so R has no multiple roots.

Since P has no multiple roots, we have the following chain of equivalent statements

- P divides Q;
- every root of P is a root of Q;
- each root of $x^{m+1} = 1$ different from 1 is not a root of $x^n = 1$;
- the system

$$\begin{cases} x^{m+1} &= 1 \\ x^n &= 1, \end{cases}$$

  has only one solution $x = 1$.

If the greatest common divisor $GCD(m+1, n) = d > 1$, then $x = \mathrm{e}^{2\pi i/d} \neq 1$ is a solution. If $GCD(m+1, n) = 1$, then we can find integers k and l such that $k(m+1) + ln = 1$. Thus every solution x satisfies

$$x = x^{k(m+1)+ln} = (x^{m+1})^k (x^n)^l = 1.$$

In summary, the answer is: P divides Q iff $m+1$ and n are relatively prime. ∎