IMC Selection Test 2

Problem 1. In how many ways can the integers from 1 to n be permuted into a sequence $(a_1, \ldots, a_n)$ such that for every $i \in \{2, \ldots, n\}$ there is $j \in \{1, \ldots, i-1\}$ with $|a_i - a_j| = 1$, that is, every element except the first differs exactly by 1 from some previous element? (For example, for $n = 3$, the sequence $(2, 1, 3)$ is allowed but $(3, 1, 2)$ is not.)

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a_0 \in \mathbb{R}$ and define the sequence (a_n) by $a_{n+1} = f(a_n)$ for all $n \geq 0$. Suppose that the sequence

$$\left(\frac{a_0 + a_1 + \cdots + a_n}{n}\right)_{n\geq 1}$$

converges. Prove that f has a fixed point (i.e. show that there exists some $c \in \mathbb{R}$ such that $f(c) = c$).

Problem 3. Determine the sequence a_r with recursive formula $\frac{(r-1)(4a_r - ra_{r-1})a_{r-2}}{a_r a_{r-1}} = 4$, where $a_0 = 2a_1 = 1$, and prove that a_r is never an integer for $r > 0$.

Problem 4. Let n be a positive integer. Evaluate the sum

$$S = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n(n-1)\ldots(n-(2r-1))}{(r!)^2}\, 2^{n-2r}.$$