# IMC Selection Test 3

**Problem 1.** Let $A$ and $B$ be positive integers such that $(1+\sqrt{7})^{2023} = A + B\sqrt{7}$. Find $\gcd(A, B)$.

**Problem 2.** An $n \times n$-matrix $A = (a_{ij})_{i,j=1}^{n}$ is *skew-symmetric* if for all $1 \le i, j \le n$ we have $a_{ij} = -a_{ji}$ (in particular, all diagonal entries are 0). Prove that if $n$ is even, then the determinant of an $n \times n$ skew-symmetric matrix $A$ does not change if we add the same number to all of its entries.

**Problem 3.** Let $f : (0, \infty) \to \mathbb{R}$ be a strictly increasing continuous function with $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to \infty} f(x) = \infty$. Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = \int_x^{2x} \frac{f(t)}{t} \mathrm{d}t$. Determine the image of $g$.

**Problem 4.** For an infinite family of compact, convex regions in the plane, the following are known:

1. every region contains at least two points;
2. every region has area $P_1 \ge 0$;
3. if two regions intersect, their intersection has area $P_2 \ge 0$;
4. there do not exist 2,023 pairwise disjoint regions in the family;
5. there does not exist a point of the plane contained in 2,023 distinct regions of the family.

Describe exactly how the regions are shaped.