IMC 2024 Training - Linear Algebra

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Below is a collection of some of the main definitions and theorems which I've found useful for solving IMC linear algebra problems. Most of these problems require nothing more than first year linear algebra and a bit of creativity!

1 Definitions

1.1 Vector spaces

(1.1.1) Let V be a vector space over F. A set of vectors $\{v_1, \ldots, v_n\} \subset V$ is

(i) **linearly independent** if the only scalars $\alpha_1, \ldots, \alpha_n \in F$ which satisfy

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

are $\alpha_1 = \cdots = \alpha_n = 0$.

(ii) **spanning** if, for all $w \in V$, there exist $\beta_1, \ldots, \beta_n \in F$ such that

$$\beta_1 v_1 + \cdots + \beta_n v_n = w.$$

(iii) a **basis** if its both linearly independent and spanning.

1.2 Orthogonality

(1.2.1) Let $v, w \in \mathbb{R}^n$. We say v is **orthogonal** to w if their dot product $v \cdot w$ is 0, i.e. if $v = (a_1, \ldots, a_n)^T$ and $w = (b_1, \ldots, b_n)^T$, then $a_1 b_1 + \cdots + a_n b_n = 0$.

More generally, for vectors $v, w \in \mathbb{C}^n$, we say v and w are orthogonal if $a_1\overline{b_1} + \cdots + a_n\overline{b_n} = 0$.

(1.2.2) Let $X \subset \mathbb{R}^n$ be a subspace. The **orthogonal complement** $X^\perp$ of X is the subspace

$$X^\perp := \{y \in \mathbb{R}^n \mid x \cdot y = 0 \quad \forall x \in X\}.$$

1.3 Matrices

(1.3.1) An $m \times n$ **matrix** is a rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

(1.3.2) The **determinant** of a square matrix A is

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

e.g. if $n = 2$, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

(1.3.3) The **trace** of a square matrix A is

$$\operatorname{tr}(A) := a_{11} + a_{22} + \cdots + a_{nn}.$$

(1.3.4) The **characteristic polynomial** of a square matrix A is $p_A(t) := \det(tI - A)$.

- E.g. For 2×2 matrices: $p_A(t) = t^2 - \operatorname{tr}(A)t + \det(A)$.

(1.3.5) The **eigenvectors** of a square matrix A are non-zero vectors v such that $Av = \lambda v$ for some scalar λ. The value λ is the **eigenvalue** associated with v.

(1.3.6) The **minimal polynomial** $\mu_A(t)$ of A is the unique monic polynomial of lowest degree such that $\mu_A(A) = 0$.

(1.3.7) The (column) **rank** of A is the maximal number of linearly independent columns of A.

1.4 Types of matrices

(1.4.1) A matrix $A = (a_{ij})$ is **diagonal** if $a_{ij} = 0$ for all i, j where $i \neq j$.

(1.4.2) A matrix A is **idempotent** if $A^2 = A$.

(1.4.3) A matrix A is **nilpotent** if $A^k = 0$ for some $k \geq 1$.

(1.4.4) A matrix A is **symmetric** if $A = A^T$.

(1.4.5) A matrix A is **Hermitian** if $A = \overline{A^T}$.

(1.4.6) A matrix A is **normal** if $A\overline{A^T} = \overline{A^T}A$.

(1.4.7) Two $n \times n$ matrices A, B are **similar** if there exists an invertible matrix P such that $A = P^{-1}BP$

(1.4.8) A matrix A is **diagonalisable** (over F) if there exists an invertible matrix $P \in F^{n \times n}$ such that $P^{-1}AP$ is diagonal.

(1.4.9) A set of matrices $A_1, \ldots, A_n$ are **simultaneously diagonalisable** if there exists an invertible matrix P such that $P^{-1}A_1P, \ldots, P^{-1}A_nP$ are all diagonal.

(1.4.10) Let $\{v_1, \ldots, v_n\}$ be n vectors in $\mathbb{R}^k$ (or $\mathbb{C}^k$). The **Gram matrix** of $\{v_1, \ldots, v_n\}$ is

$$G(\{v_1, \ldots, v_n\}) = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \ldots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \ldots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \ldots & \langle v_n, v_n \rangle \end{pmatrix}$$

where $\langle v_i, v_j \rangle$ is the usual inner product on $\mathbb{R}^k$ (or $\mathbb{C}^k$).

2 Useful facts

2.1 Bases

- Every vector space has a basis.
- All bases of V have the same cardinality. This is the **dimension** of V.
- Any linearly independent set can be extended to a basis.
- Any spanning set can be reduced to a basis.
- If V is a vector space of dimension n, then the following are equivalent for a set of n vectors $S = \{v_1, \ldots, v_n\}$:

$$S \text{ is linearly independent} \iff S \text{ spans } V \iff S \text{ is a basis for } V.$$

2.2 Orthogonality

- (*Gram-Schmidt*) Every vector space (with an inner product) has an orthonormal basis.
- Any set of non-zero pairwise orthogonal vectors is a linearly independent set.
- Let X be subspace of $\mathbb{R}^n$, then $\mathbb{R}^n = X \oplus X^\perp$ (namely, $\dim(X) + \dim(X^\perp) = n$)

2.3 Determinants

- Determinants are well-behaved with respect to row or column operations:
  - (i) Multiplying a row/column by c multiples the determinant by c.
  - (ii) Swapping two rows/columns multiples the determinant by -1.
  - (iii) Adding a scalar multiple of one row to another row does not change the determinant.
- The determinant is homogeneous: $\det(cA) = c^n \det(A)$.
- The determinant is multiplicative: $\det(AB) = \det(A)\det(B)$.
- A matrix A is invertible if and only if $\det(A) \neq 0$.
- Transpositions: $\det(A^T) = \det(A)$.
- If A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity), then $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.
- If A is triangular, then $\det(A) = a_{11} a_{22} \cdots a_{nn}$.
- Determinants can be calculated recursively: Let $M_{i,j}$ be the determinant of the $(n-1) \times (n-1)$ matrix that is formed by removing the ith row and jth column from A. Then

$$\det(A) = (-1)^{i+1} a_{i1} M_{i1} + (-1)^{i+2} a_{i2} M_{i2} + \cdots + (-1)^{i+n} a_{in} M_{in}$$

  for any $i = 1, \ldots, n$.

2.4 Trace

- The trace is linear: $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$
- The trace is cyclic: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- Transpositions: $\mathrm{tr}(A) = \mathrm{tr}(A^T)$
- If A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity), then $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
- More generally, $\mathrm{tr}(A^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$ (counted with multiplicity).

2.5 Rank

- The column rank of a matrix equals its row rank.
- An square $n \times n$ matrix A is invertible if and only if $\mathrm{rank}(A) = n$.
- Decomposition: $\mathrm{rank}(A)$ is the smallest k such that there exists an $m \times k$ matrix C, and an $k \times n$ matrix R such that $A = CR$.
- Largest minor: $\mathrm{rank}(A)$ is the largest k such that there is a $k \times k$ submatrix of A with non-zero determinant.
- Number of non-zero eigenvalues of A is at most $\mathrm{rank}(A)$.
- Row operations do not change the rank.
- The rank is sub-additive: $\mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$.

2.6 Diagonalisation

- A matrix is diagonalisable over F if and only if its minimal polynomial is a product of distinct linear factors over F.
- An $n \times n$ matrix is diagonalisable over F if and only if it has n linearly independent eigenvectors (in F^n).
- An $n \times n$ matrix A is diagonalisable over F if it has n distinct eigenvalues in F (but converse *not* true!)
- Any real symmetric matrix is diagonalisable (over $\mathbb{R}$).
- The only nilpotent diagonalisable matrix is the zero matrix.
- A set of diagonalisable matrices commutes if and only if the set is simultaneously diagonalisable.

2.7 Cayley-Hamilton Theorem

- Let A be an $n \times n$ matrix with characteristic polynomial $p(x) := \det(xI - A)$. Then $p(A) = 0$. (i.e. the minimal polynomial divides the characteristic polynomial).

2.8 Gram matrices

- Any Gram matrix is positive semidefinite (i.e. Hermitian with nonnegative eigenvalues)
- Vectors $\{v_1, \ldots, v_n\}$ are linearly independent if and only if Gram determinant is non-zero.
- Rank of Gram matrix equals the rank of the $k \times n$ matrix formed by $\{v_1, \ldots, v_n\}$.

2.9 Miscellaneous

- Any eigenvalue λ_i of a matrix A is a root of both the characteristic polynomial $p_A(t)$ and the minimal polynomial $\mu_A(t)$.
- For any two square matrices A, B, we have $p_{AB}(t) = p_{BA}(t)$. Furthermore, if at least one of A, B is invertible then $\mu_{AB}(t) = \mu_{BA}(t)$.
- If two $n \times n$ matrices A and B are similar, then $p_A(t) = p_B(t)$ and $\mu_A(t) = \mu_B(t)$. If $n \leq 3$, then converse is also true.
- Any real symmetric matrix (or Hermitian matrix) has all real eigenvalues.
- A matrix is nilpotent if and only if all its eigenvalues are 0.
- For any idempotent matrix A, $\mathrm{tr}(A) = \mathrm{rank}(A)$.

3 IMC Practice Problems

Below are a list of 60 IMC linear algebra problems, given very roughly in order of difficulty. This should hopefully keep you busy until the competition in August :). Feel free to drop me an email if you're stuck and would like a hint!

1. Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

2. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

3. Let V be a 10-dimensional real vector space and U_1 and U_2 two linear subspaces such that $U_1 \subseteq U_2$, $\dim_{\mathbb{R}} U_1 = 3$ and $\dim_{\mathbb{R}} U_2 = 6$. Let E be the set of all linear maps $T : V \to V$ which have U_1 and U_2 as invariant subspaces (i.e. $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$). Calculate the dimension of E as a real vector space.

4. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\text{trace}(M) = a$ and $\det(M) = b$.

5. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation $A^{-1} + B^{-1} = (A + B)^{-1}$ prove that $\det(A) = \det(B)$. Does the same conclusion follow for matrices with complex entries?

6. (a) Show that for any $m \in \mathbb{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix.
   (b) Show that $\det A > 0$ for every real $m \times m$ matrix satisfying $A^3 = A + I$.

7. Let A be the $n \times n$ matrix, whose (i, j)-th entry is $i + j$ for all $i, j = 1, 2, \ldots, n$. What is the rank of A?

8. Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left-to-right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

9. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

10. Let A be a real $n \times n$ matrix such that $A^3 = 0$.
    (a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation $X + AX + XA^2 = A$.
    (b) Express X in terms of A.

11. (a) Let A be a $n \times n$, $n \geq 2$, symmetric invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.
    (b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2 & 2 & \ldots & 2 \\ 1 & 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \ldots & \ddots \end{pmatrix}?$$

12. Let V be a real vector space, and let $f, f_1, f_2, \ldots, f_k$ be linear maps from V to $\mathbb{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \cdots = f_k(x) = 0$. Prove that f is a linear combination of $f_1, f_2, \ldots, f_k$.

13. Let X be a nonsingular matrix with columns $X_1, X_2, \ldots, X_n$. Let Y be a matrix with columns $X_2, X_3, \ldots, X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0's for eigenvalues.

14. Let A be a 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u \in \mathbb{R}^3$. Prove that:

    (a) $A^T = -A$, where A^T denotes the transpose of A;

    (b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$, where $v \times u$ denotes the vector product in $\mathbb{R}^3$.

15. Let $a_j = a_0 + jd$ for $j = 0, \ldots, n$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}$$

    Calculate $\det(A)$.

16. Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if } i \neq j \\ 2, & \text{if } i = j. \end{cases}$$

17. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

    Find BA.

18. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

19. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that $2021B = A^m + B^2$. Prove that $|\det A| \leq 1$.

20. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

    for some integer $k \geq n$.

21. Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

22. Let M be an invertible matrix of dimension $2n \times 2n$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Show that $\det M \cdot \det H = \det A$.

23. Let A and B be $n \times n$ real matrices such that $\mathrm{rk}(AB - BA + I) = 1$. Prove that

$$\mathrm{trace}(ABAB) - \mathrm{trace}(A^2B^2) = \frac{1}{2}n(n-1).$$

24. A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx + Ey + Az + Rw &= Y \\ Rx + Yy + Ez + Aw &= E \\ Ax + Ry + Yz + Ew &= A \\ Ex + Ay + Rz + Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

25. Let k and n be positive integers. A sequence $(A_1, \dots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

26. Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii}a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j$$

and determine all matrices for which equality holds.

27. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

28. Does there exist a real 3×3 matrix A such that $\mathrm{tr}(A) = 0$ and $A^2 + A^T = I$?

29. Let A, B and C be $n \times n$ matrices with complex entries satisfying $A^2 = B^2 = C^2$ and $B^3 = ABC + 2I$. Prove that $A^6 = I$.

30. Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A - B)C = BA^{-1}$, then $C(A - B) = A^{-1}B$.

31. Let $A_1, A_2, \dots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_iA_j = -A_jA_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

32. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

33. Define the sequence $A_1, A_2, \dots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \dots)$$

where I_m is the $m \times m$ identity matrix. Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \dots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$, respectively.

34. Let $A, B \in M_n(\mathbb{C})$ be two $n \times n$ matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

35. Let n be a positive integer, and consider the matrix $A = (a_{ij})_{1 \le i,j \le n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

36. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1\dots n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Find $\det(A)$.

37. Let A be an $n \times n$-matrix with integer entries and $b_1, \dots, b_k$ be integers satisfying $\det A = b_1 \cdot \dots \cdot b_k$. Prove that there exist $n \times n$-matrices $B_1, \dots, B_k$ with integer entries such that $A = B_1 \cdot \dots \cdot B_k$ and $\det B_i = b_i$ for all $i = 1, \dots, k$.

38. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \dots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \le i, j \le n+1$. Prove that $v_1, \dots, v_{n+1}$ are linearly dependent over the rationals.

39. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V \quad \operatorname{trace}(XY) = 0$$

40. For $n \ge 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ with multiplicities $m_1, m_2, \dots, m_k$ respectively. Consider the linear operator L_M defined by $L_M(X) = MX + XM^T$, for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities.

41. Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$. Show that the sequence A^k converges to an idempotent matrix.

42. Let A be an $n \times n$ complex matrix such that $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

43. Let A, B be square complex matrices of the same size such that

$$\mathrm{rank}(AB - BA) = 1$$

Show that $(AB - BA)^2 = 0$.

44. Let $\alpha \in \mathbb{R}\backslash\{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^2$ satisfying $F \circ G - G \circ F = \alpha F$.

    (a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

    (b) Show that there exists $k \geq 1$ such that $F^k = 0$.

45. Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \dots (x - c_k)^{d_k}$$

where $c_1, c_2, \dots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diaognal, etc. and $d_1 + d_2 + \dots d_k = n$).

Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is $d_1^2 + d_2^2 + \cdots + d_k^2$.

46. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

47. Let A be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

48. Let A and B be real $n \times n$ matrices. Assume that there exist $n + 1$ different real numbers $t_1, t_2, \dots, t_{n+1}$ such that the matrices

$$C_i = A + t_i B, \quad i = 1, 2, \dots, n + 1$$

are nilpotent (i.e. $C_i^n = 0$). Show that both A and B are nilpotent.

49. The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $\dim V = n < \infty$.

    (i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

    (ii) Find the maximal number of distinct pairwise commuting involutions on V.

50. Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is an invertible matrix then n is divisible by 3.

51. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

52. (a) Let $f : M_n \to \mathbb{R}$ be a linear map from the space $M_n = \mathbb{R}^{n^2}$ of real $n \times n$ matrices to the reals. Prove that there exists a unique matrix $C \in M_n$ such that $f(A) = \operatorname{tr}(AC)$ for any $A \in M_n$.
(b) Suppose in addition to (a) that $f(AB) = f(BA)$ for any $A, B \in M_n$. Prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda \cdot \operatorname{tr}(A)$.

53. An $n \times n$ complex matrix A is called *t-normal* if $AA^T = A^T A$ where A^T is the transpose of A. For each n, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of *t-normal* matrices.

54. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2} \|A\|^{n-1}.$$

(here $\|B\| = \sup_{\|x\| \leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^n |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

55. Let A be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \iff \exists S \in \mathrm{GL}_n(\mathbb{C}) \text{ such that } A = S\overline{S}^{-1}.$$

56. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

(a) $\det(B) = 1$;
(b) $AB = BA$;
(c) $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

57. For each positive integer k, find the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \ldots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \cdots = A_k^2 = 0$,
(2) $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq k$, and
(3) $A_1 A_2 \ldots A_k \neq 0$.

58. For an $m \times m$ real matrix A, e^A is defined as $\sum_{n=0}^{\infty} \frac{1}{n!} A^n$. (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent.

59. For $n \geq 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$ for every $n, k \geq 1$.

60. Let A_i, B_i, S_i $(i = 1, 2, 3)$ be invertible real 2×2 matrices such that

(1) not all A_i have a common real eigenvector;
(2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;
(3) $A_1 A_2 A_3 = B_1 B_2 B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.