IMC 2024 Training
Expectation is linear

Problem 1 (Warmup)**.**

- A dice is rolled n times, what is the expected number of different sides that come up among these n rolls?
- A coin is tossed n times, with the results recorded in a sequence (e.g. HHTHTTH). What is the expected number of consecutive HHH in the result sequence?

Problem 2*(Putnam 2023)**.** A sequence of real numbers $(x_1, \cdots, x_k)$ is a zigzag sequence if the numbers $x_2 - x_1, \cdots, x_k - x_{k-1}$ are all non-zero and alternate in signs. Given a permutation $\sigma \in S_n$, let $X(\sigma)$ be the maximum possible k such that there exists $1 \leq i_1 < \cdots < i_k \leq n$ with $(\sigma(i_1), \cdots, \sigma(i_k))$ being a zigzag sequence. What is the expected value of $X(\sigma)$ if $\sigma \in S_n$ is chosen uniformly at random?

Problem 3 (Coupon collector's problem)**.** There are n types of coupons, and every box of cereal contains a uniformly random type of coupon inside. What is the expected number of boxes of cereal one needs to buy to collect all n types coupons?

Problem 4*(Putnam 2022)**.** Suppose $X_1, X_2, \cdots$ are real numbers in $[0, 1]$ chosen independently and uniformly at random. Let k be the smallest positive integer such that $X_k < X_{k+1}$, where we let $k = \infty$ if no such k exists. Find the expected value of $S = \sum_{i=1}^{k} X_i$.

Problem 5 (ICMC 2024 Round Two)**.** Let N be a fixed positive integer, let S be the set $\{1, 2, ..., N\}$, and let F be the set of functions $f : S \to S$ such that $f(i) \geq i$ for all $i \in S$. For each $f \in F$, let P_f be the unique polynomial of degree less than N satisfying $P_f(i) = f(i)$ for all $i \in S$. If f is chosen uniformly at random from F, determine the expected value of $(P_f)'(0)$.

Problem 6*(Putnam 2016)**.** Let A be a $2n \times 2n$ matrix with each of its entries chosen independently and uniformly at random between 0 and 1. Find the expected value of $\det(A - A^T)$.

Problem 7*(Online Math Open 2013 Fall)**.** Consider the following procedure starting with all 2^n subsets of $\{1, 2, \cdots, n\}$. In each round, one of the remaining subsets, say S, is picked uniformly at random, and then the set S, along with all remaining sets that are subsets of S are removed. What is the expected number of rounds that this procedure will last?

Problem 8***.** Show that the randomised quicksort algorithm has expected runtime $O(n \log n)$.

Problem 9$^*_*$(IMC 2022)**.** Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

Problem 10$^*_*$(Putnam 2004)**.** Each square of an $m \times n$ checkerboard is coloured independently at random with red or black with equal probability $\frac{1}{2}$. We say that two squares p and q are in the same connected monochromatic region if there is a sequence of squares, all of the same colour, starting with p and ending with q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is at least $\frac{1}{8}mn$.