Solutions for problems on convergence and limits

Problem 1 Solution: For convenience, denote $a_n := (6n+1)/(n+1)$. First we show by mathematical induction that the sequence (s_n) is non-decreasing. Note that $s_0 = 1 < s_1 = (9/2)^{1/3}$. Assume that $s_{n-1} \le s_n$. Note that $a_n < a_{n+1}$. Thus we have

$$s_n = (a_n + s_{n-1})^{1/3} \le (a_{n+1} + s_n)^{1/3} = s_{n+1}.$$

Next we show, also by induction, that the sequence (s_n) is bounded above by 2. Clearly, $s_0 = 1 < 2$. Assume that $s_{n-1} < 2$. Then we have

$$s_n = (a_n + s_{n-1})^{1/3} < (6+2)^{1/3} = 2.$$

Thus $L := \lim_{n\to\infty} s_n$ exists. Letting $n \to \infty$ in $s_n^3 = a_n + s_{n-1}$, we obtain $L^3 = 6 + L$. The roots are 2 and $-1 \pm i\sqrt{2}$. Since each s_n is a real number, the limit is 2. ∎

Problem 2 Solution: No: note $n^{1+1/n} = ne^{\log n/n} = (1+o(1))n$.

Alternatively, it is easy to see that $n < 2^n$ for every $n \ge 1$. Hence $n^{1/n} < 2$, so

$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}.$$

Since $\sum 1/(2n)$ is divergent, so is our sum. ∎

Problem 3 Solution: No. Suppose the contrary. Then for all large $n \ge n_0$ we have $|a_i - \alpha| < \alpha/4$. This implies that the ratio of a_n/a_{n+1} cannot be larger than $(1+1/4)/(1-1/4) = 5/3 < 2$. This means that $a_{n+1} = \sqrt{a_n}$ for all $n \ge n_0$. Then α satisfies $\alpha = \sqrt{\alpha}$, a contradiction. ∎

Problem 4 Solution: Since $\sqrt{x}$ is monotone increasing function, we have

$$1 + \frac{m^{1/2}-1}{1/2} = 1 + \int_1^m \frac{dx}{\sqrt{x}} \le 1 + \sum_{n=2}^m n^{-1/2} \le \sum_{n=1}^m n^{-1/2} \le \int_1^{m+1} \frac{dx}{\sqrt{x}} = \frac{(m+1)^{1/2}-1}{1/2}.$$

It follows that the limit in question is 2. ∎

Problem 5 Solution: No, it is not. Clearly, the summands are decreasing with n. It is enough to show that the sum

$$\sum_{n=10^7}^{\infty} \frac{1}{n \times \ln n \times \ln(\ln n)} \tag{1}$$

is not converget, since it dominates the original sum. Now observe that the indefinite integral

$$\int \frac{1}{x \times \ln x \times \ln(\ln x)} = \ln(\ln(\ln(x))) + C,$$

approaches ∞ as $x \to \infty$, so the sum (1) does not converge.

Alternatively, this problem can be solve by splitting the summation range into intervals $[2^k+1, 2^{k+1}]$ and bounding the sum over the k-th interval by

$$2^k \times \frac{(\log_2 e)^2}{2^{k+1} k \ln\ln\ln k} \le \frac{2}{k \ln k}.$$

But we proved in class that the sum $\sum \frac{1}{k \ln k}$ is divergent. Hence, our sum is divergent. ∎