Practice problems for the session on Combinatorics

These problems provide some practice for the ideas/concepts discussed at the session. They are not a part of selection for Warwick's IMC team. However, you are very welcome to try to solve them and then check your solutions, using the sample solutions from the website or by coming to my office hours in B2.12 Zeeman, 1:30-2:30pm on Thursdays, except 30 January when I am away. *–Oleg Pikhurko*

Problem 1 What is c_n, the maximum number of regions that n circles in the plane can create?

Problem 2 What is s_n, the maximum number of regions that n spheres in $\mathbb{R}^3$ can create?

Problem 3 What is t_n, the maximum number of regions in $\mathbb{R}^2$ that can be created by n triangles such that each triangle has $(0, 0)$ on its boundary?

Problem 4 Given real numbers $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, 3, 4$, such that $a_1b_2 - a_2b_1 \neq 0$. Consider the set of all solutions $(x_1, \ldots, x_4)$ of the simultaneous equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= 0, \end{aligned}$$

for which no x_i $(i = 1, 2, 3, 4)$ is zero. Each such solution generates a 4-tuple of plus and minus signs. What is the maximum possible number of distinct 4-tuples of signs?

What is the answer if we allow nonhomogeneous linear equations:

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= c_1, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= c_2 \end{aligned}$$

(with the same restriction that $a_1b_2 - a_2b_1 \neq 0$)?