Solutions to practice problems

Read these solutions only after you have seriously attempted the problems!

Problem 1 Solution: We have $c_1 = 2$. We can assume that no two circle are the same since otherwise we can move one of them arbitrarily without decreasing the number of regions. Add circles one by one. When we add the i-th circle then it has at most $2(i-1)$ points of intersection with the earlier circles and thus cuts at most $2(i-1)$ regions. We conclude that $c_i \le c_{i-1} + 2(i-1)$. Thus

$$c_n \le 2 + \sum_{i=2}^{n} 2(i-1) = 2 + 2\binom{n}{2} = n^2 - n + 2.$$

This bound is sharp: take any n circles such that every two intersect in two points and no three have a common point, for example, the rotations around $(0,0)$ by n distinct small angles of the circle $\{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1\}$. ∎

Problem 2 Solution: We have $s_1 = 2$. Let $i \ge 2$. To upper bound the number of new regions that the i-th sphere S can create, we have to upper bound the maximum number of regions on the surface of S that $i-1$ other spheres can create. This can be bounded by the parameter c_i from the previous problem because in proving an upper bound on c_i we only use the fact that each two distinct circles intersect in at most two points (which is still true for circles on a sphere, where we can eliminate any collection of coinciding circles by perturbing arbitrarily all but one of them). Thus

$$s_i \le s_{i-1} + c_{i-1} = s_{i-1} + (i-1)^2 - (i-1) + 2 = s_{i-1} + i^2 - 3i + 4.$$

This is sharp: if we take n spheres in general position (every two intersect in a circle, no three having a common circle, no 4 having a common point) then each of the above inequality becomes equality.

We conclude that

$$s_i = 2 + \sum_{j=2}^{i} (j^2 - 3j + 4).$$

Suppose that we do not remember the formula for $\sum_{j=k}^{i} \binom{j}{k}$ but we remember that it is a polynomial in i of degree $k+1$. So the sum $2 + \sum_{j=2}^{i}(j^2 - 3j + 4)$ is some polynomial $Q(i) = f_3 i^3 + \ldots + f_0$ and we have to determine its coefficients. We do this one by one, starting with the highest-degree one. (We could have also started from the lowest-degree one.) We have

$$j^2 - 3j + 4 = Q(i) - Q(i-1) = f_3 \cdot 3i^2 + \text{terms of degree at most 2}.$$

It follows that $f_3 = 1/3$. Let $Q_2(i) = Q(i) - i^3/3$. We have

$$Q_2(i) - Q_2(i-1) = i^2 - 3i + 4 - i^3/3 + (i-1)^3/3 = -2i + 11/3.$$

Thus $f_2 = -1$. Let $Q_1 = Q_2 + i^2$. Then

$$Q_1(i) - Q_2(i) = (-2i + 11/3) + i^2 - (i-1)^2 = 8/3,$$

so $Q_1(i) = 8i/3 + f_0$ and $Q = i^3/3 - i^2 + 8i/3 + f_0$. Finally, $Q(1) = 2$, so $1/3 - 1 + 8/3 + f_0 = 2$ or $f_0 = 0$. Thus

$$s_i = i^3/3 - i^2 + 8i/3.$$

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Problem 3 Solution: Clearly, $t_1 = 2$. One can show that the intersection of the i-th triangle with the previous $i-1$ triangles consists of at most $5(i-1)+1$ (contiguous) pieces, because each triangle contains the origin $(0,0)$. If we remove $5(i-1)+1$ (contiguous) pieces from a closed curve then we get $5(i-1)+1$ pieces and this upper bounds the number of the regions cut by the i-th triangle. Hence $t_i \le t_{i-1} + 5(i-1) + 1$. We conclude that

$$t_i \le 2 + 5i(i-1)/2 + (i-1) = 5i^2/2 - 3i^2/2 + 1.$$

This inequality it sharp since we can fix a triangle T containing $(0,0)$ inside its boundary (but not as one of its 3 vertices) and take small rotations of T by n different angles around $(0,0)$ for the n triangles. In this configuration, every two triangles' boundaries intersect in 6 points, while no point apart of $(0,0)$ can belong to the boundaries of more than two of these triangles. ∎

Problem 4 Solution: Each choice of (x_3, x_4) gives the (unique) pair (x_1, x_2) via some linear functions $x_1 = L_1(x_3, x_4)$ and $x_2 = L_2(x_3, x_4)$. We can assume that neither L_1 nor L_2 is not identically zero for otherwise we have no sign sequences at all.

The coordinate axis and the lines $L_1 = 0$ and $L_2 = 0$, all coming through $(0,0)$, partition the plane into at most 8 "sign" regions. This is best possible: for example, take two equations $x_1 + x_3 + x_4 = 0$ and $x_2 + x_3 - x_4 = 0$ (when the lines $L_1 = 0$ and $L_2 = 0$ go at angles $\pm\pi/2$).

Note that if we look at an inhomogeneous system

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= c_1, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= c_2, \end{aligned}$$

then the corresponding lines $L_1 = 0$ and $L_2 = 0$ need not pass through the origin so the number of possible sign sequences is at most the maximum number of regions defined by any 4 lines, which is 11. Again, this is best possible. Take two lines $L_1 = 0$ and $L_2 = 0$ that, with the coordinate axes, define 11 regions; then let the two equations be $x_1 - L_1(x_3, x_4) = 0$ and $x_2 - L_2(x_3, x_4) = 0$. ∎