IMC SEMINAR
Selection Test 1
Solutions

Problem 1 *Let $a > 0$ and let $f(x)$ be a continuos function on $[0, a]$ such that $f(x) > 0$ and $f(x)f(a - x) = 1$ for every $x \in [0, a]$. Evaluate*

$$\int_0^a \frac{dx}{1 + f(x)}.$$

Solution. Clearly, $0 < 1/(1 + f(x)) < 1$ so the integral exists. Setting $x = a - y$ we obtain

$$\begin{aligned} I &= \int_0^a \frac{dx}{1+f(x)} = \int_a^0 \frac{-dy}{1+f(a-y)} \\ &= \int_0^a \frac{dy}{1+f(a-y)} = \int_0^a \frac{f(x)dy}{f(y)+1} \\ &= \int_0^a \left(1 - \frac{1}{1+f(y)}\right) dy = a - I. \end{aligned}$$

Hence, $I = a/2$.

Problem 2 *Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.*

Solution. Choose a set of a rows $r_1, \ldots, r_a$ containing an $a \times b$ submatrix whose entries are all 1. Then for $i, j \in \{1, \ldots, a\}$, we have $r_i \cdot r_j = n$ if $i = j$ and 0 otherwise. Hence

$$\sum_{i,j=1}^{a} r_i \cdot r_j = an.$$

On the other hand, the term on the left is the dot product of $r_1 + \cdots + r_a$ with itself, i.e., its squared length. Since this vector has a in each of its first b coordinates, the dot product is at least $a^2 b$. Hence $an \geq a^2 b$, whence $n \geq ab$ as desired.