24/Feb/2015

IMC SEMINAR
Selection Test 2
Solutions

Problem 1 *Let $x = 0.001002004008016032064128\ldots$ be the real number in the interval $(0,1)$ whose decimal expression is obtained by concatenating the last three digits of the sequence of powers of 2 (padded by 0's in case there is only one or two digits). Is x rational? Explain.*

Solution. There are finitely many reminders mod 1000, in particular the sequence of powers of 2 mod 1000 is finite. Moreover, once a repetition is reached the sequence then becomes periodic. This is exactly the same as saying that the decimal expansion of x is periodic. Therefore x is rational. ∎

Problem 2 *Compute the eigenvalues of the $n \times n$ matrix*

$$P = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$$

Solution. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors. Then $P\mathbf{e}_i = \mathbf{e}_{i+1}$ with index taken mod n, so $P^n = I$. Since I has eigenvalue 1 with multiplicity n, each eigenvalue of P is a root of $x^n = 1$. We claim that all n roots are eigenvalues. By the dimension consideration, it enough to exhibit an eigenvector for each root ω^s, $s \in [n]$, where $\omega = \exp(2\pi i/n)$. We take $\mathbf{u}_s = \sum_{k=1}^n \omega^{ks}\mathbf{e}_k$. We have

$$P\mathbf{u}_s = \sum \omega^{ks} P\mathbf{e}_k = \sum \omega^{ks}\mathbf{e}_{k+1} = \omega^{-s} \sum (\omega^{s(k+1)}\mathbf{e}_{k+1}) = \omega^{-s} u_s,$$

as desired. ∎