Selection Tests

Please note that the IMC selection tests will take place 1-2pm on the following dates:

- 17 Feb in B3.02;
- 24 Feb in B3.02;
- 9 Mar in MS.04.

Homework problems (due at the test on 17 February)

Problem 1 Let X be a finite set and $A_1, \ldots, A_{50}$ be its subsets such that each A_i has strictly more than half of elements of X. Prove that there is $B \subset X$ of size at most 5 such that B intersects every A_i.

Problem 2 Let a set X of size 2^n be partitioned into some subsets $A_1, \ldots, A_m$. We can repeat the following operation: for some sets A_i and A_j with $|A_i| \geq |A_j|$, move $|A_j|$ elements from A_i to A_j. (Thus the size of A_j doubles.) Prove that starting from any initial partition we can make one set to be equal to the whole set X.

Problem 3 Suppose that sets $A_1, \ldots A_m$, each of size r, satisfy that $|A_i \cap A_j| \leq k$ for all $1 \leq i < j \leq m$. Prove that

$$|A_1 \cup \ldots \cup A_m| \geq \frac{m\, r^2}{k(m-1)+r}.$$

Problem 4 Prove that every graph $G = (V, E)$ (with no loops allowed) admits a vertex partition $V = V_1 \cup V_2$ such that, for both $i = 1, 2$, all degrees in the induced subgraph $G[V_i]$ are even. (In other words, we require that every vertex $x \in V$ had the even number of neighbours in its part.)