Probability

A discrete probability space Ω is a countable set imbued with a probability function $\mathbb{P} : \Omega \to [0, 1]$ such that $\sum_{x\in\Omega} \mathbb{P}(x) = 1$. A random variable is a function $X : \Omega \to \mathbb{R}$

Given a discrete random variable $X : \Omega \to S \subset \mathbb{R}$, the expectation $\mathbb{E}(X)$ is equal to $\sum_{s\in S} s\mathbb{P}(X = s)$. The variance $\mathbb{V}(X)$ is equal to $\mathbb{E}((X - \mathbb{E}(X))^2)$

Given an event $A \subset \Omega$, its indicator function $I_A : \Omega \to \{0, 1\}$ is defined to be 1 on A and 0 on $\Omega \backslash A$.

Theorem (Markov's inequality): For $t > 0$ and a positive random variable $X > 0$ we have $\mathbb{P}(X > t) < \frac{\mathbb{E}(X)}{t}$

Theorem: For a positive random variable $X > 0$ we have $\mathbb{P}(X = 0) \leq \frac{\mathbb{V}(X)}{\mathbb{E}(X)^2}$

The probability generating function of a random variable X is defined to be $G_X(z) \equiv \sum_{s\in S} z^s\mathbb{P}(z = s)$

Problems

Problem 1: Show that a graph G has a bipartition $V(G) = V_1 \cup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{e(G)}{2}$ (where $e(G[U])$ means the number of edges whose end points are both in U)

Problem 2: Given s, t integers, show that for every $n \geq 1$, there is a bipartite graph with both parts of size n , at least $\frac{1}{2}n^{2-\frac{s+t-2}{st-1}}$ edges, but doesn't contain a $K_{s,t}$ (the complete bipartite graph with partitions of sizes s and t)

2012/1/1: For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For example, $p(4) = 5$ because:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n - 1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1

2016/2/4 Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, x_2, ...x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + |x_2| + ... + |x_k| \leq n$. Prove that for every $n \geq 1$, $f(n - 1)f(n + 1) \leq f(n)^2$

2016/1/5 Let S_n denote the set of permutations of the sequence $(1, 2, ..., n)$. For every permutation $\pi = (\pi_1, \pi_2, ...p_n) \in S_n$, let $inv(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$, ie the number of inversions in π. Denote by $f(n)$ the number of permutations of $\pi \in S_n$ for which $inv(\pi)$ is divisible by $n + 1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$ and infinitely manu primes p for which $f(p - 1) < \frac{(p-1)!}{p}$.