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IMC problem solving seminar

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Analysis

1. The definite integrals between 0 and 1 of the squares of the continuous real functions $f(x)$ and $g(x)$ are both equal to 1. Prove that there is a real number c such that $f(c) + g(c) \leq 2$.

2. Find all continuously differentiable functions $f : [0, 1] \to (0, \infty)$ such that $\frac{f(1)}{f(0)} = e$ and

$$\int\limits_0^1 \frac{\mathrm{d}x}{f^2(x)} + \int\limits_0^1 f'^2(x)\,\mathrm{d}x \leq 2.$$

Number Theory

3. The positive divisors of a positive integer n are written in increasing order starting with 1: $1 = d_1 < d_2 < d_3 < \cdots < n$. Find n if it is known that:

- $n = d_{13} + d_{14} + d_{15}$
- $(d_5 + 1)^3 = d_{15} + 1$

4. Let $d(k)$ denote the number of all natural divisors of a natural number k. Prove that for any natural number n_0 the sequence $\{d(n^2 + 1)\}_{n=n_0}^{\infty}$ is not strictly monotone.

Homework

5. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a continuous and periodic function. Prove that for all $\alpha \in \mathbb{R}$ the following inequality holds:

$$\int_0^T \frac{f(x)}{f(x+\alpha)} dx \geq T,$$

where T is the period of $f(x)$.

6. Given a nonincreasing differentiable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ prove that

$$\int\limits_0^\infty e^{-t-f(t)} \sqrt{1 + (f'(t))^2}\,\mathrm{d}t \geq \sqrt{\alpha^2 + (\alpha - e^{-f(0)})^2},$$

where $\alpha = \int\limits_0^\infty e^{-t-f(t)}\,\mathrm{d}t.$

7. Let $f : \mathbb{N} \to \mathbb{R}$ be given by $f(n) = n^{\frac{\tau(n)}{2}}$ where $\tau(n)$ - is the number of divisors of n. Show that f is injective into $\mathbb{N}$.

8. *Integer numbers $x > 2, y > 1, z$ are such that $x^y + 1 = z^2$. Let p denote the number of different divisors of x and q - the number of different divisors of y. Prove that $p \geq q + 2$