Problem 1. What is the maximum value of

$$\sum_{i=1}^{n} |\sigma(2i) - \sigma(2i-1)|$$

over all permutations σ of $\{1, \ldots, 2n\}$?

Problem 2. Let $a > 0$ and let $f(x)$ be a continuous function on $[0, a]$ such that $f(x) > 0$ and $f(x)f(a-x) = 1$ for every $x \in [0, a]$. Evaluate

$$\int_0^a \frac{dx}{1 + f(x)}.$$

Problem 3. For every positive integer n, let a_n be the number of terms in the sequence $2^1, 2^2, \cdots, 2^n$ whose base-10 representation begins with the digit 1. Find $\lim_{n\to\infty} a_n/n$.

Problem 4. Recall a complex number α is an *n-th primitive root of unity* if n is the smallest positive integer such that $\alpha^n = 1$. The *cyclotomic polynomial* $\phi_n(z)$ is the product of $z - \alpha$ over all primitive n-th roots of unity α.

1. Prove that $z^n - 1 = \prod_{d|n} \phi_d(z)$, where the product is over all $d \in \{1, \ldots, n\}$ that divide n.

2. For each integer $n \geq 1$, determine the value of $\phi_n(1)$.

Problem 5. Let n, m be positive integers and let $A_1, \ldots, A_m$ be subsets of $\{1, 2, \ldots, n\}$. For each non-empty $S \subseteq \{1, 2, \ldots, m\}$, let $A_S := \cup_{k \in S} A_k$. Define a function f on $[0, 1]$ by

$$f(x) := \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|-1} x^{|A_S|}, \quad x \in [0, 1].$$

Show that f is non-decreasing.