Solution to Problem 1. The maximum is n^2. One of many examples attaining it is when σ has respectively values $1,\ldots,n$ on $1,3,5,\ldots,2n-1$ and $n+1,n+2,\ldots,2n$ on $2,4,6,\ldots,2n$: then we add n terms, each equal to n.

On the other hand, given σ let us define

$$\begin{aligned} a_i &:= \max(\sigma(2i),\sigma(2i-1)),\\ b_i &:= \min(\sigma(2i),\sigma(2i-1)), \quad \text{for } 1\le i\le n. \end{aligned}$$

Then $a_1,\ldots,a_n,b_1,\ldots,b_n$ is a permutation of $1,\ldots,2n$ and thus

$$\begin{aligned} \sum_{i=1}^n |\sigma(2i)-\sigma(2i-1)| &= \sum_{i=1}^n (a_i-b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i\\ &\le ((n+1)+\ldots+2n)-(1+\ldots+n) = n^2, \end{aligned}$$

as desired.

Solution to Problem 2. Clearly, $0<1/(1+f(x))<1$ and the integrated function is continuous. Thus the integral exists. Setting $x=a-y$ we obtain

$$\begin{aligned} I &= \int_0^a \frac{dx}{1+f(x)} = \int_a^0 \frac{-dy}{1+f(a-y)}\\ &= \int_0^a \frac{dy}{1+f(a-y)} = \int_0^a \frac{f(y)dy}{f(y)+1}\\ &= \int_0^a \left(1-\frac{1}{1+f(y)}\right) dy = a-I. \end{aligned}$$

Hence, $I=a/2$.

Solution to Problem 3. We show that $\lim_{n\to\infty} a_n/n = \log_{10} 2$. It suffices to show $a_n = \lfloor n\log_{10} 2\rfloor$ for all positive integer n, as then

$$\log_{10} 2 - \frac{1}{n} \le \frac{a_n}{n} \le \log_{10} 2.$$

Fix a positive integer n and let $m := \lfloor n\log_{10} 2\rfloor$. The main idea of the proof is in brief that every integer between 2^1 and 2^n that starts with 1 belongs to exactly one of the m intervals $[10^k, 2\cdot 10^k)$, $1\le k\le m$, while each interval contains precisely one power of 2. More formally, for each $i\in[n]$, 2^i starts with the digit 1 if and only if $10^k\le 2^i<2\cdot 10^k$ for some positive integer k. Since $10^{m+1} > 10^{n\log_{10}2} = 2^n$, we must have $k\le m$. On the other hand, we claim that for every $k\in[m]$, there exists a unique $i\in[n]$ satisfying $10^k\le 2^i < 2\cdot 10^k$. Indeed, fix a $k\in[m]$ and let $i\in[n]$ be minimal subject to $10^k\le 2^i$. Such i exists as $10^k\le 10^m\le 10^{n\log_{10}2}=2^n$. Also, $i\ge 4$ as $10^k\ge 10>2^3$. We claim that this is the unique i that works. By minimality of i, $2^j<10^k$ for all $j<i$. In particular, $2^{i-1}<10^k$, and thus $10^k\le 2^i<2\cdot 10^k$. Meanwhile, for all $j>i$, $2^j\ge 2\cdot 2^i\ge 2\cdot 10^k$.

Alternatively, observe that 2^i starts with digit 1 if and only if $10^{\lfloor \log_{10} 2^i\rfloor} \le 2^i < 2\cdot 10^{\lfloor \log_{10} 2^i\rfloor}$. This is equivalent to $\{i\log_{10}2\}\in[0,\log_{10}2)$, where $\{i\log_{10}2\}$ denotes the fractional part of $i\log_{10}2$. Since $\log_{10}2$ is irrational, by the Equidistribution Theorem, as n tends to infinity, the proportion of $i\in[n]$ satisfying $\{i\log_{10}2\}\in[0,\log_{10}2)$ is $\log_{10}2$.

Solution to Problem 4. 1. The polynomial $z^n - 1$ factorises as the product of $z - \alpha$ over all n-th roots of unity α. For every such α, its *order* m (the smallest integer $m \geq 1$ such that $\alpha^m = 1$) is a divisor of n and the product of $z - \alpha$ over all order-m roots α is exactly $\phi_{n/m}(z)$.

2. Let us prove by induction on n that

$$\phi_n(1) = \begin{cases} 0, & n = 1, \\ p, & n = p^k,\ k \in \mathbb{N},\ p \text{ is prime}, \\ 1, & \text{otherwise}. \end{cases}$$

This is true if $n = 1$ as $\phi_1(z) = z - 1$. If $n = p$ is prime, then $\phi_p(z)$ is the product of $z - \alpha$ over all p-th roots of unity different from 1. Thus $\phi_p(z) = \frac{z^p - 1}{z - 1} = z^{p-1} + \cdots + 1$ and $\phi_p(1) = p$. Suppose that $n \geq 2$ is not a prime. Let the prime factorisation of n is $\prod_{i=1}^s p_i^{m_i}$ for some distinct primes p_i with each $m_i \geq 1$. By the identity in the first part divided by $\phi_1(z) = z - 1$, we know that

$$z^{n-1} + \cdots + 1 = \phi_n(z) \prod_{\substack{d|n \\ 1<d<n}} \phi_d(z),$$

If we substitute $z = 1$ then we get that

$$n = \phi_n(1) \prod_{\substack{0 \leq k_1 \leq m_1,\ \cdots,\ 0 \leq k_s \leq m_s, \\ (k_1,\ldots,k_s) \neq (0,\ldots,0) \text{ or } (m_1,\ldots,m_s)}} \phi_{p_1^{k_1} \ldots p_s^{k_s}}(1).$$

The last term is different from 1 only if all k's are zero except exactly one k_i is between 1 and m_i, when the value is p_i by induction. If $s \geq 2$ then there are exactly m_i choices of k_i for each i and thus $n = \phi_n(1) = \prod_{i=1}^n p_i^{m_i}$ and $\phi_n(1) = 1$. If $s = 1$, then there are exactly $m_1 - 1$ choices for k_1 (as we now have to avoid $k_1 = m_1$), so $p_1^{m_1} = \phi_n(1) = p_1^{m_1 - 1}$ and $\phi_n(1) = p_1$, as desired.

Solution to Problem 5. For each $A \subseteq [n]$ and $x \in [0, 1]$, let $R_A(x) \subseteq [0, 1)^n$ be given by

$$R_A(x) := \{(t_1, \ldots, t_n) \in [0, 1)^n \mid t_i < x \text{ for all } i \in A\}.$$

Then $\text{vol}(R_A(x)) = x^{|A|}$. Note that for all $\emptyset \neq S \subseteq [m]$, $R_{A_S}(x) = \cap_{k \in S} R_{A_k}(x)$. Hence, we have, by the Inclusion-Exclusion Principle,

$$\begin{aligned} f(x) &= \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|-1} \text{vol}(R_{A_s}(x)) \\ &= \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|-1} \text{vol}(\cap_{k \in S} R_{A_k}(x)) \\ &= \text{vol}(\cup_{k=1}^m R_{A_k}(x)). \end{aligned}$$

Hence, it suffices to show that for all $0 \leq x \leq y \leq 1$, we have $\cup_{k=1}^m R_{A_k}(x) \subseteq \cup_{k=1}^m R_{A_k}(y)$. Indeed, if $(t_1, \ldots, t_n) \in \cup_{k=1}^m R_{A_k}(x)$, then there exists $\ell \in [m]$ such that $t_i < x$ for all

$i \in A_\ell$. It follows that $t_i < y$ for all $i \in A_\ell$, and thus $(t_1, \ldots, t_n) \in R_{A_\ell}(y) \subseteq \cup_{k=1}^m R_{A_k}(y)$, as required.

Alternatively, let $B_1, \cdots, B_n$ be independent random events each occuring with probability x. For every $k \in [m]$, let E_k be the event that B_i occurs for all $i \in A_k$. For every $\emptyset \neq S \subset [m]$, let E_S be the event that E_k occurs for all $k \in S$, or equivalently B_i occurs for all $i \in \cup_{k \in S} A_k = A_S$. Then $\mathbb{P}(E_S) = x^{|A_S|}$, and by a similar application of the Inclusion-Exclusion Principle, we have $f(x) = \mathbb{P}(\cup_{k=1}^m E_k)$, which is evidently non-decreasing.