Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and

$$f(x) + f(f(x)) = 2024, \quad \text{for every } x \in \mathbb{R}.$$

Determine $f(1000)$ and give an example of such a function f.

Problem 2. Find all polynomials $P(x)$ with real coefficients for which there exists a non-zero polynomial $Q(x)$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$P(x)Q(x) = \sum_{i=0}^{n} a_i x^{2^i},$$

where $n = \deg(P)$ is the degree of P.

Problem 3. Let non-negative reals $a_1, a_2, \ldots, a_7, b_1, b_2, \ldots, b_7$ be such that

$$a_i + b_i \leq 2, \quad \text{for each} \quad i = 1, 2, \ldots, 7.$$

Prove that there are two distinct numbers $k, m \in \{1, 2, \ldots, 7\}$ such that

$$|a_k - a_m| + |b_k - b_m| \leq 1.$$

Problem 4. Consider the sequence defined by the formula

$$a_n = \frac{1}{1 + \cos(n\pi\sqrt{2})}.$$

Determine if the sequence $\left(\frac{a_n}{n^2}\right)_{n=1}^{\infty}$ is bounded or not.

Problem 5. Let A be a random $n \times n$ matrix, each of whose entries is chosen independently and uniformly at random to be either -1 or 1. Let $X = \det(A)$ be the determinant of A. Compute its variance $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.