Solution to Problem 1. Using $f(0) = 1$, we get $1 + f(1) = 2024$, so $f(1) = 2023$. Since f is continuous, there exists $y \in (0, 1)$ such that $f(y) = 1000$. It follows that $1000 + f(1000) = 2024$, so $f(1000) = 1024$.

An example of such f is

$$f(x) = \begin{cases} 1, \text{ if } x \in (-\infty, 0] \cup [2023, \infty), \\ 2022x + 1, \text{ if } x \in (0, 1), \\ 2024 - x, \text{ if } x \in (1, 2023). \end{cases}$$

Solution to Problem 2. Answer: $P(x)$ can be any polynomial.

(Short Solution) Let V be the subspace of the vector space of polynomials with basis x^{2^i}, $i = 0, \ldots, n$. Let $W \subset \mathbb{R}[x]$ be the ideal generated by $P(x)$. Then there is a natural projection $\pi : \mathbb{R}(x) \to \mathbb{R}(x)/W$ which is a linear map. Viewing π as map from $V \to \mathbb{R}(x)/W$, we see that the kernel $\ker(\pi)$ must be non trivial since $\dim(V) = n + 1$, but $\dim(\mathbb{R}(x)/W) = n$. If $R \neq 0$ is in the kernel, then $R = PQ$ for some $Q \neq 0$ (by $\pi(R) = 0$) and non-zero coefficients in R are restricted to x^{2^i} for $0 \leq i \leq n$ (by $R \in V$).

(Long Solution) Consider the quotients of x^{2^i} by $P(x)$, $i = 0, 1, \ldots n$. We can write these down as

$$\begin{aligned} x^1 &= P(x)S_0(x) + R_0(x) \\ x^2 &= P(x)S_1(x) + R_1(x) \\ x^4 &= P(x)S_2(x) + R_2(x) \\ &\vdots \\ x^{2^n} &= P(x)S_n(x) + R_n(x), \end{aligned}$$

where $\deg(R_i(x)) < n$. Sine the polynomials R_i live in an n-dimensional vector space spanned by $0, x, \ldots, x^{n-1}$ and since there are $n+1$ of them, they are linearly dependent. Thus there exist $a_0, a_1, \ldots, a_n \in \mathbb{R}$, not all 0, such that

$$\sum_{i=0}^{n} a_i R_i(x) = 0$$

Consider then

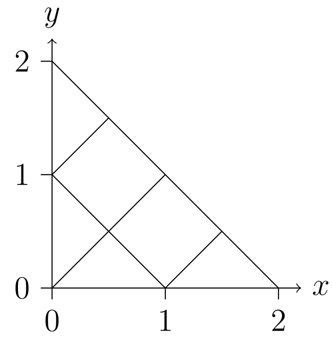
$$\begin{aligned} a_0 x^1 &= a_0 P(x)S_0(x) + a_0 R_0(x) \\ a_1 x^2 &= a_1 P(x)S_1(x) + a_1 R_1(x) \\ a_2 x^4 &= a_2 P(x)S_2(x) + a_2 R_2(x) \\ &\vdots \\ a_n x^{2^n} &= a_n P(x)S_n(x) + a_n R_n(x), \end{aligned}$$

which we may sum to obtain

$$\sum_{i=0}^{n} a_i x^{2^i} = P(x) \sum_{i=0}^{n} a_i S_i(x) + \sum_{i=0}^{n} a_i R_i(x) = P(x)Q(x),$$

where $Q(x) = \sum_{i=0}^{n} a_i S_i(x)$.

Solution to Problem 3. We consider 7 points in the (x, y)-plane with coordinates (a_i, b_i). The condition $a_i + b_i \le 2$ implies that these all lie in a triangle bound by x and y axis (since a_i, b_i are non negative) and the line $y = 2 - x$. We may split this triangle into six regions as in the image below.



So one of these regions contains two points. It is now easy to verify that two points in the same regions will satisfy the condition.

Solution to Problem 4. This sequence can only be unbounded if for any $C > 0$, $a_n > Cn^2$ for some n. We may transform this condition into:

$$1 + \cos(n\pi\sqrt{2}) < \frac{1}{Cn^2}$$

We note that cosine is close to -1 only when the argument is near and odd integer multiple p of π so we expand it using Taylor's series around $p\pi$:

$$1 + \cos(p\pi + (p\pi - n\pi\sqrt{2})) = (p\pi - n\pi\sqrt{2})^2/2 + O((p\pi - n\pi\sqrt{2})^4)$$

By absorbing some terms into the constant this reduces to showing that for any constant $C > 0$ there is some p, n for which

$$(p - n\sqrt{2})^2 < \frac{1}{Cn^2}$$

or

$$\left|\frac{p}{n} - \sqrt{2}\right| < \frac{1}{Cn^2}$$

We show that this is impossible for, say, $C = 10$. Suppose that there was such a pair p, n. Since p is odd, we have that

$$\left|\left(\frac{p}{n}\right)^2 - 2\right| = \left|\frac{p^2 - 2n^2}{n^2}\right| \ge \frac{1}{n^2}.$$

Thus

$$\left(\frac{p}{n}\right)^2 - 2 = \left(\frac{p}{n} - \sqrt{2}\right)\left(\frac{p}{n} + \sqrt{2}\right) < \frac{2\sqrt{2} + 1}{10n^2} < \frac{1}{n^2},$$

which is a contradiction.

Solution to Problem 5. Answer: $\mathrm{Var}[X] = n!$.

We have that

$$\begin{aligned}\mathbb{E}[\det(A)] &= \mathbb{E}\left[\sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\prod_{i=1}^{n}A_{i,\sigma(i)}\right]\\ &= \sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\prod_{i=1}^{n}\mathbb{E}\left[A_{i,\sigma(i)}\right] \;=\; 0,\end{aligned}$$

where we used that the entries of A are mutually independent.

Furthermore, we have that

$$\begin{aligned}\mathbb{E}[\det(A^2)] = \mathbb{E}[\det(A)^2] &= \mathbb{E}\left[\left(\sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\prod_{i=1}^{n}A_{i,\sigma(i)}\right)^2\right]\\ &= \sum_{\sigma\in S_n}\sum_{\tau\in S_n}(-1)^{\mathrm{sign}(\sigma)+\mathrm{sign}(\tau)}\mathbb{E}\left[\prod_{i=1}^{n}A_{i,\sigma(i)}A_{i,\tau(i)}\right]\\ &= \sum_{\sigma\in S_n}\sum_{\tau\in S_n}(-1)^{\mathrm{sign}(\sigma)+\mathrm{sign}(\tau)}\prod_{i=1}^{n}\mathbb{E}\left[A_{i,\sigma(i)}A_{i,\tau(i)}\right],\end{aligned}$$

where the last equality follows as for distinct $i\in[n]$, $A_{i,\sigma(i)}A_{i,\tau(i)}$ are independent variables. Using independence again, we see that $\mathbb{E}\left[A_{i,\sigma(i)}A_{i,\tau(i)}\right] = \mathbb{E}\left[A_{i,\sigma(i)}\right]\mathbb{E}\left[A_{i,\tau(i)}\right] = 0$ unless $\sigma(i)=\tau(i)$, in which case it is equal to 1. Hence, the product above is non-zero if and only if $\sigma=\tau$, in which case it is equal to 1. Therefore,

$$\mathbb{E}[\det(A^2)] = \sum_{\sigma\in S_n}(-1)^{2\cdot\mathrm{sign}(\sigma)}\cdot 1 = n!.$$