Problem 1. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of positive integers. Suppose that $f : \mathbb{N} \to \mathbb{N}$ satisfies that $f(1) = 1$ and, for all $n \in \mathbb{N}$,

a) $3f(n)f(2n+1) = f(2n)(3f(n)+1)$,

b) $f(2n) < 6f(n)$.

Show that this determines f uniquely. Find all solutions to the equation $f(k) + f(m) = 2026$ in positive integers $k < m$.

Problem 2. Let A be a real $n \times n$ matrix with rank 1, and let I be the $n \times n$ identity matrix. Prove that
$$\det(I + A^2) \geq 2\operatorname{tr}(A)$$

Problem 3. Let $S = \{1, 2, \dots, 2n\}$. Let A be a subset of S such that $|A| \geq n+1$. Prove that A contains two distinct elements x, y such that x/y is an integer.

Problem 4. Evaluate
$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$