

**Solution to Problem 1.** Since  $3f(n)$  and  $3f(n) + 1$  are relatively prime, a) implies that  $f(2n)$  is divisible by  $3f(n)$ . This combined with b) gives  $f(2n) = 3f(n)$ . Using a) again, we have  $f(2n + 1) = 3f(n) + 1$ . Thus  $f$  is unique.

One can guess (and easily prove by induction) that that  $f$  acts in the following way. Write  $n$  in base 2, and let  $f(n)$  have the same digits but in base 3. Eg  $f(7) = f(111_2) = 111_3 = 13$ .

The number of solutions to  $f(k) + f(m) = 2026$  is the same as the number of way to write  $2026 = 2210001_3$  as a sum of two numbers whose ternary digits are 0 and 1. Note that when two such digits are added, there no digit is carried over. So we have freedom in a binary digit of  $k$  and  $m$  only when the corresponding ternary digit of 2026 is 1 (and then we have 2 choices). By  $k < m$ , we must have in binary that  $k = (110000b)_2$  and  $m = (111000b')_3$  with  $\{b, b'\} = \{0, 1\}$ . Thus all solutions are  $(k, m) = (96, 113)$  and  $(97, 112)$ .

**Solution to Problem 2.** We may choose a basis of  $A$  for which it has the following form

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

where the first basis vector is some non-zero vector in  $\text{im}(A)$ , and the other ones are chosen as arbitrary basis vectors of  $\text{ker}(A)$ . With that we see that

$$\det(I + A^2) \geq 2 \text{tr}(A)$$

becomes

$$\begin{aligned} 1 + a_1^2 &\geq 2a_1 \\ (a_1 - 1)^2 &\geq 0 \end{aligned}$$

**Solution to Problem 3.** Let  $S = \{1, 2, \dots, 2n\}$  and let  $A \subseteq S$  be a subset such that  $|A| \geq n + 1$ .

Every positive integer  $k$  can be uniquely written in the form  $k = 2^p \cdot m$ , where  $p \geq 0$  is an integer and  $m$  is an odd integer. We refer to  $m$  as the odd part of  $k$ .

For any element  $k \in S$ , its odd part  $m$  must be an odd integer in the range  $[1, 2n]$ . The set of all possible odd parts is  $O = \{1, 3, 5, \dots, 2n - 1\}$ . The size of this set is  $|O| = n$ . We map each element  $a \in A$  to its corresponding odd part  $m$ .

Since  $|A| \geq n + 1$  and there are only  $n$  odd parts, by the Pigeonhole Principle, at least two distinct elements of  $A$  must be mapped to the same odd part.

Let  $x \in A$  and  $y \in A$ , with  $x \neq y$ , be two such elements. These two elements share the same odd part  $m$ . We can thus write:

$$\begin{aligned} x &= 2^k \cdot m \\ y &= 2^j \cdot m \end{aligned}$$

for some integers  $k, j \geq 0$ .

Since  $x \neq y$ , it must be that  $k \neq j$ . Assume, without loss of generality, that  $k > j$ . We then compute the ratio  $x/y$ :

$$\frac{x}{y} = \frac{2^k \cdot m}{2^j \cdot m} = 2^{k-j}$$

Because  $k > j$ , the exponent  $k - j$  is a positive integer. Therefore,  $x/y$  is an integer.

**Solution to Problem 4.** Let us add the series to itself, swapping the role of  $n$  and  $m$  in the second series and rearranging:

$$2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n 3^n + n^2 m 3^m}{3^m 3^n (n 3^m + m 3^n)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{3^m 3^n} = \left( \sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2.$$

These rearrangements are legal since all terms are positive.

How do we compute the last sum? One way is to note that  $\sum_{n=1}^{\infty} n/3^n = f(1)$ , where  $f(z) = \sum_{n=1}^{\infty} n z^n / 3^n$ , with the radius of convergence of this power series being 3. We have

$$f(z) = z \sum_{n=0}^{\infty} (z^n / 3^n)' = z(1 + z/3 + z^2/3^2 + \dots)' = z \left( \frac{1}{1 - z/3} \right)', \quad |z| < 3.$$

(The power series  $\sum_{n=0}^{\infty} (z/3)^n$  also has the radius of convergence 3 so we can differentiate it term by term for  $|z| < 3$ .) Hence,  $f(z) = (1 - z/3)^{-2} \cdot (z/3)$  and  $f(1) = (2/3)^{-2} / 3 = 3/4$ . Thus  $S = (3/4)^2 / 2 = 9/32$ .