Solution to Problem 1. This problem is from the 2008 IMC.

Fix $q \in \mathbb{Q}$. The function

$$x \longmapsto f(x+q) - f(x)$$

is continuous and takes values in $\mathbb{Q}$. Hence it must be constant. Therefore, there exists a function $g : \mathbb{Q} \to \mathbb{Q}$ such that $f(x+q) - f(x) = g(q)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q}$.

Clearly, g is additive, so $g(q) = cq$ for all $q \in \mathbb{Q}$, where $c = g(1) \in \mathbb{Q}$. Thus

$$f(x+q) = f(x) + cq.$$

In particular, taking $x = 0$, we obtain $f(q) = f(0) + cq$ for every $q \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous, it follows that $f(x) = cx + b$ for all $x \in \mathbb{R}$, where $c \in \mathbb{Q}$ and $b = f(0) \in \mathbb{R}$. Conversely, every function of this form clearly satisfies the required condition. Hence the answer is

$$\boxed{f(x) = cx + b, \quad c \in \mathbb{Q},\ b \in \mathbb{R}.}$$

Solution to Problem 2. This is Putnam Problem B5 (1984).

Define

$$D(x) := (1-x)(1-x^2)(1-x^4)\dots(1-x^{2^{m-1}}).$$

Since binary expansions are unique, each monomial x^k $(0 \le k \le 2^m - 1)$ appears exactly once in the expansion of $D(x)$, with coefficient $(-1)^{d(k)}$. That is

$$D(x) = \sum_{k=0}^{2^m-1} (-1)^{d(k)} x^k.$$

Applying the operator $x\frac{d}{dx}$ to $D(x)$ m times, we obtain $\sum_{k=0}^{2^m-1}(-1)^{d(k)}k^m x^k$. We are interested in the value of this expression for $x = 1$. Define $F(x) = D(x+1)$. We have, as $x \to 0$,

$$F(x) = \prod_{a=1}^{m}(1-(x+1)^{2^{a-1}}) = \prod_{a=1}^{m}(-2^{a-1}x + O(x^2)) = Ax^m + O(x^{m+1}),$$

where $A = (-1)^m 2^{m(m-1)/2}$. Observe that $[(x+1)d/dx](x^n) = nx^n + nx^{n-1}$, we see that

$$\left[(x+1)\frac{d}{dx}\right]^m (Ax^m + O(x^{m+1})) = m!A + O(x).$$

Hence the answer is $m!A = (-1)^m 2^{m(m-1)/2} m!$.

Solution to Problem 3. This is Problem 932 from the book *"Putman and Beyond"*.

The angle at the vertex P_j is acute if and only if all other points lie on an open semicircle facing P_j. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Otherwise, the two arcs that these angles subtend would overlap and cover the whole circle, and the sum of the measures of the two angles would exceed $180°$.

So the polygon has either just one acute angle or two consecutive acute angles. In particular, taken in counter-clockwise order, there exists exactly one pair of consecutive angles the second of which is acute and the first of which is not.

We are left with the computation of the probability of this for one of the points P_j, the angle at P_j is not acute, but the following angle is. This can be done using integrals. But there is a clever argument that reduces the geometric probability to a probability with a finite number of outcomes. The idea is to choose randomly $n-1$ pairs of antipodal points, and then among these to choose the vertices of the polygon. A polygon with one vertex at P_j and the other among those points has the desired property exactly when $n-2$ vertices lie on the semicircle to the clockwise side of P_j and one vertex on the opposite semicircle. Moreover, the points on the semicircle should include the counterclockwise-most to guarantee that the angle at P_j is not acute. Hence there are $n-2$ favorable choices out of the total 2^{n-1} choices of points from the antipodal pairs. The probability for obtaining a polygon with the desired property is therefore $(n-2)2^{-n+1}$.

Integrating over all choices of pairs of antipodal points preserves the ratio. The events $j = 1, 2, \ldots, n$ are independent, so the probability has to be multiplied by n. The answer to the problem is therefore $n(n-2)2^{-n+1}$.

Solution to Problem 4. This problem is from the 2011 IMC.

Let $K = \mathbb{F}_p$ and $h(x) = x^p - x + 1 \in K[x]$. We first show that h is irreducible over K. Let α be a root of h in an algebraic closure of K. Since $\alpha^p = \alpha - 1$, we obtain $\alpha^{p^j} = \alpha - j$ for every nonnegative integer j. Thus the first p elements of the Frobenius orbit of α are distinct, and $\alpha^{p^p} = \alpha$. Hence the minimal polynomial of α over K has degree p, and so h is irreducible.

(a) Since $\alpha^{p^p} = \alpha$ and $\alpha \neq 0$, we have $\alpha^{p^p-1} = 1$. The same holds for every root of $h(x)$, since they are all of the form $\alpha - j$ for $j \in \mathbb{F}_p$. Moreover, these roots are distinct. Therefore, every root of $h(x)$ is a root of $x^{p^p-1} - 1$, and hence

$$h(x) \mid x^{p^p-1} - 1 \quad \text{in } \mathbb{F}_p[x].$$

Thus $p^p - 1$ is interesting.

(b) Set $m = 1 + p + \cdots + p^{p-1} = \frac{p^p-1}{p-1}$. Using $\alpha^{p^j} = \alpha - j$, we obtain

$$\alpha^m = \prod_{j=0}^{p-1} \alpha^{p^j} = \prod_{j=0}^{p-1} (\alpha - j).$$

Since the elements $\alpha - j$, for $j = 0, \ldots, p-1$, are precisely the roots of $h(x) = x^p - x + 1$, comparing constant terms gives $\alpha^m = (-1)^p$.

If p is odd, then $\alpha^m = -1$, and hence $\alpha^{2m} = 1$. Thus $2m$ is interesting. For $p \geq 5$, we have $2m = \frac{2(p^p-1)}{p-1} < p^p - 1$. Therefore $p^p - 1$ is not the smallest interesting number for any prime $p \geq 5$.

If $p = 2$, since $\alpha \neq 1$ and $\alpha^3 = 1$, the multiplicative order of α is 3. Hence $2^2 - 1 = 3$ is the smallest interesting number.

If $p = 3$, then $m = 1 + 3 + 9 = 13$, so

$$\alpha^{13} = -1, \qquad \alpha^{26} = 1.$$

Thus the order of α divides 26 but does not divide 13. Since 13 is prime, the only possibilities are 2 and 26. However, $\alpha \neq -1$, because

$$h(-1) = (-1)^3 - (-1) + 1 = 1 \neq 0 \quad \text{in } \mathbb{F}_3.$$

Hence the order of α is 26. Therefore $3^3 - 1 = 26$ is the smallest interesting number.

We conclude that

$$\boxed{p \in \{2, 3\}}.$$