Real analysis, exercises.

Solved in the class:

1. (IMC 2009, day 1)

$f, g\colon \mathbb{R} \to \mathbb{R}$ are such that $f(r) \leq g(r)$ for all $r \in \mathbb{Q}$. Is it true that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$ if

(a) f, g are nondecreasing,

(b) f, g are continuous?

2. (IMC 2007, day 2)

$C \subset \mathbb{R}$ is nonempty, closed and bounded, and $f\colon C \to C$ is continuous and nondecreasing. Show that there exists $p \in C$ such that $f(p) = p$.

3. (IMC 2015, day 2)

Compute

$$\lim_{A\to\infty} \frac{1}{A} \int_1^A A^{1/x} \mathrm{d}x.$$

4. (IMC 2010, day 1)

Let $0 < a < b$. Show that

$$\int_a^b (x^2+1)\mathrm{e}^{-x^2} \mathrm{d}x \geq \mathrm{e}^{-a^2} - \mathrm{e}^{-b^2}.$$

For self-study:

5.

Show that if f is twice differentiable on $\mathbb{R}$ with $f(0) = 0$ then there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)\left(1 + 2\tan^2(\xi)\right).$$

Hint: use the mean value theorem twice, to $g(x) := f(x)\cos x$ *and then to* $h(x) := g'(x)/\cos^2 x$.

6.

Let $f \in \mathbb{R} \to (0, \infty)$ be continuously differentiable. Prove that there exists $\xi \in (0, 1)$ such that

$$\mathrm{e}^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}.$$

7.

Let all C^1 functions $f\colon [0, 1] \to (0, \infty)$ such that

$$\frac{f(1)}{f(0)} = \mathrm{e} \qquad \text{and} \quad \int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x \leq 2.$$

Hint: $(x + 1/x)^2 = x^2 + 1/x^2 + 2$.

8.

Does there exist an injective function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - f(x)^2 \geq \frac{1}{4}?$$

Hint: when $x^2 = x$*?*

Wojciech Ozanski, Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK.
E-mail address: W.S.Ozanski@warwick.ac.uk