Polylogarithmic Eisenstein classes and special values of L-functions

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Classical Eisenstein series

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where $\zeta: \mathbb{C} \smallsetminus \{1\} \to \mathbb{C}$ is the Riemann zeta function given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } s \in \mathbb{C}, \mathrm{Re}(s) > 1.$$

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$$\widetilde{\mathrm{Sym}^k\,\mathbb{C}^2}(U) = \left\{ \begin{array}{c} \phi{:}\pi^{-1}(U)\to\mathrm{Sym}^k\,\mathbb{C}^2 \text{ locally constant:} \\ \phi(\gamma\tau)=\gamma(\phi(\tau)) \text{ for all } \tau\in\pi^{-1}(U),\gamma\in\mathrm{SL}_2(\mathbb{Z}) \end{array} \right\}$$

for open $U \subseteq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\pi\colon \mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is the projection.

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The assignment $f \mapsto \delta_f$ induces a Hecke-equivariant isomorphism

$$\mathbb{M}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \oplus \overline{\mathbb{S}}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \cong H^1(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}).$$

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Observe that

$$\mathbb{M}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{S}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \oplus \mathbb{C} \cdot G_{k+2}.$$

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Eisenstein classes and Eisenstein cohomology

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$$\mathbb{M}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \oplus \overline{\mathbb{S}}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \cong H^1(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}).$$

$$\mathbb{S}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \oplus \overline{\mathbb{S}}_{k+2}(\mathrm{SL}_2(\mathbb{Z})) \overset{\text{E.–S.}}{\longleftrightarrow} \ker(R^*).$$

- We call $\omega \in H^1(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ an *Eisenstein class* if

$$\omega \notin \ker(R^*).$$

- The *Eisenstein cohomology* is the subspace

$$H^1_{\mathrm{Eis}}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) := \mathbb{C} \cdot G_{k+2}.$$

$$H^1(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \cong \ker(R^*) \oplus H^1_{\mathrm{Eis}}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$$

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Consequence: If $\omega \in H^1(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}, \widetilde{\mathrm{Sym}^k\mathbb{C}^2})$ lies in the image of $H^1(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}, \widetilde{\mathrm{Sym}^k\mathbb{Q}^2})$,

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for open $U \subseteq T$, where $\pi\colon \mathbb{R}^n \to T$ is the projection.

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$\rightsquigarrow$ *topological polylogarithm* $\mathrm{pol}_{\alpha} \in H^{n-1}(T \smallsetminus D, \Gamma; \mathscr{L}\mathrm{og}_{\mathbb{Q}} \otimes \lambda)$.

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Polylogarithmic Eisenstein classes for SL_2

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$\rightsquigarrow$ Obtain a cohomology class

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$$\widetilde{\mathscr{G}}_\alpha^k(g) = \sum_{\mu \in \mathbb{Z}^2 \setminus \{0\}} E_\mu^k(g) \Big(\sum_{t \in T[M]} \alpha(t) e^{2\pi i \mu(t)} \Big),$$

where

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$$\cdot \Big(\mu_1 gg^t \mu^t \cdot d(\mu_2 gg^t \mu^t) - \mu_2 gg^t \mu^t \cdot d(\mu_1 gg^t \mu^t) \Big)$$

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Explicit calculation: If $\Gamma = SL_2(\mathbb{Z})$, then

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Adam's theorem: If $p-1 \nmid k+2$, then

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