Polylogarithmic Eisenstein classes and special values of L-functions

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Abstract

One striking feature of special values of L-functions is that they appear as constant terms of Eisenstein series. The goal of this article is to use this fact to motivate the definition of Eisenstein (cohomology) classes. Further, we shall discuss how such classes may be constructed via the topological polylogarithm, and how this yields insights into special values of L-functions.

Note: The most recent version of this article may be found at https://sites.google.com/view/lukas-prader/mathematics

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1 Classical Eisenstein series and their constant terms

Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts (from the left) on the upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} \colon \mathrm{Im}(\tau) > 0\} \subseteq \mathbb{C}$$

by Möbius transformations, i.e., if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, then

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

For any $k \in \mathbb{Z}$, this gives rise to a right action $|_k$ on the space of all functions $f \colon \mathbb{H} \to \mathbb{C}$ defined by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau).$$

A *(holomorphic) modular form for* $\mathrm{SL}_2(\mathbb{Z})$ *of weight* $k \in \mathbb{Z}$ is a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ satisfying the following two conditions:

(i) $f|_k\gamma = f$ holds for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

In particular, letting $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this implies that f is 1-periodic.

(ii) f is regular at infinity, i.e., in the Fourier series expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n\tau}$$

of f (which exists by (i)), we have $a_n = 0$ for every $n < 0$.

We call a_0 the *constant term* of f. It may alternatively be characterized[1] by

$$a_0 = \lim_{y \to \infty} f(x + iy) \tag{1}$$

for any $x \in \mathbb{R}$. If $a_0 = 0$, then we say that f is a *cusp form*.

A simple way of constructing modular forms is by means of *Eisenstein series*. Indeed, for any $k \geq 1$, the Eisenstein series[2]

$$G_{k+2}(\tau) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m\tau + n)^{k+2}} \tag{2}$$

is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $k + 2$. However, condition (i) applied to $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ reads as $G_{k+2}(\tau) = (-1)^{k+2} G_{k+2}(\tau)$, suggesting that

[1] This comes from the fact that the Fourier series is normally convergent on $\mathbb{H}$; see [4, Prop. III.5.4].

[2] The reader might find it strange that we consider G_{k+2} for $k \geq 1$ instead of G_k for $k \geq 3$. Indeed, this has a cohomological reason, which will become apparent in §2.

$G_{k+2} = 0$ for odd k. On the other hand, if k is even, then the Fourier series expansion of G_{k+2} is given by

$$G_{k+2}(\tau) = 2\zeta(k+2) + \frac{2(2\pi i)^{k+2}}{(k+1)!} \sum_{n=1}^{\infty} \big(\sum_{d|n} d^{k+1}\big) e^{2\pi i n \tau}, \tag{3}$$

cf. [4, Prop. VII.1.3], where ζ denotes the *Riemann zeta function* $\zeta\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$, which for $s \in \mathbb{C}, \mathrm{Re}(s) > 1$ may be described by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In view of (1), this implies that G_{k+2} is not the zero function, hence gives a non-trivial example of a modular form.

Since any constant multiple of G_{k+2} will again be a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $k+2$, we may normalize G_{k+2} in such a way that its n-th Fourier coefficient for $n \geq 1$ is given by $\sum_{d|n} d^{k+1}$. It turns out that the resulting Eisenstein series, which we shall denote by G^*_{k+2}, has Fourier series

$$G^*_{k+2}(\tau) = 2^{-1}\zeta(-1-k) + \sum_{n=1}^{\infty} \big(\sum_{d|n} d^{k+1}\big) e^{2\pi i n \tau}, \tag{4}$$

where we used that, by the functional equation for the Riemann zeta function,

$$\zeta(k+2) = \frac{(2\pi i)^{k+2}}{2(k+1)!}\zeta(-1-k). \tag{5}$$

This provokes the following question, which we shall take as the starting point for our mathematical journey:

What does it mean for $\zeta(-1-k)$ to appear in the constant term of an Eisenstein series, all of whose remaining Fourier coefficients are rational numbers?

More precisely, can we use the plenty of symmetries inherent in G^*_{k+2} (in the sense of (i)) to draw any conclusions on the arithmetic nature of the real number $\zeta(-1-k)$? Indeed, this is the underlying idea of Klingen's proof[3] [10] that $\zeta(-1-k) \in \mathbb{Q}$ for every even $k \geq 0$.
In the current text, however, we shall take a cohomological approach to the above question (motivated by the Eichler–Shimura isomorphism (10)), which will enable us (using the method from [1]) to provide quite sharp bounds[4] for the denominator of $\zeta(-1-k)$. One major difference between our approach and Klingen's work is that we also have a cohomological notion of rationality (and integrality), which is detached from the rationality of Fourier coefficients.

[3]Note that Klingen's proof actually works for a wide class of zeta functions, including Dedekind's zeta function associated to a totally real number field.

[4]The zeta values $\zeta(-1-k)$ may be expressed in terms of Bernoulli numbers (as will be explained in §9), which yields an explicit prime factorization for the denominator of $\zeta(-1-k)$. This allows us to check the quality of our results.

But before moving on, in order to enrich our presentation, we shall briefly introduce a more general type of modular form.
To this aim, we introduce for any $N \in \mathbb{N}$ the *principal congruence subgroup*

$$\Gamma[N] = \ker(\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

of level N. A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma \supseteq \Gamma[N]$ for some $N \in \mathbb{N}$, and if N is minimal with this property, we say that Γ has *level* N. As a consequence, congruence subgroups $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ are of finite index.
Now a *(holomorphic) modular form for* Γ *of weight* $k \in \mathbb{Z}$ is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the following two conditions:

(i') $f|_k\gamma = f$ holds for every $\gamma \in \Gamma$.
In particular, letting $\gamma = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma[N] \subseteq \Gamma$, this implies that f is N-periodic.

(ii') For every $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the function $f|_k\alpha\colon \mathbb{H} \to \mathbb{C}$ is regular at infinity (in the sense of (ii)).

Concerning the second condition, a few remarks are in order. Let $\{\alpha_1, \ldots, \alpha_r\} \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a system of representatives[5] for $\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})$, i.e.,

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^{r} \Gamma\alpha_j, \tag{6}$$

then (ii') may be replaced by the equivalent condition

(ii") For every $1 \leq j \leq r$, the function $f|_k\alpha_j\colon \mathbb{H} \to \mathbb{C}$ is regular at infinity.

Since $\Gamma[N] \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a normal subgroup (being a kernel), each $f|_k\alpha_j$ is a modular form for $\Gamma[N]$ (but not necessarily for Γ), thus N-periodic. So (ii") precisely says that for each $1 \leq j \leq r$ we have a Fourier series expansion

$$(f|_k\alpha_j)(\tau) = \sum_{n=0}^{\infty} a_{n,j} e^{2\pi i n\tau/N}.$$

In analogy with the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we call $a_{0,1}, \ldots, a_{0,r}$ the *constant terms* of f, and say that f is a *cusp form* if $a_{0,1} = \ldots = a_{0,r} = 0$. Finally, we denote by $\mathbb{M}_k(\Gamma)$ (respectively $\mathbb{S}_k(\Gamma)$) the $\mathbb{C}$-vector space of (holomorphic) modular forms (respectively cusp forms) for Γ of weight $k \in \mathbb{Z}$.

For a detailed discussion of modular forms, we shall refer to the excellent textbooks [4], [11] and [12]. Further information on Riemann's zeta function may be found in [3].

[5] If we were talking about fundamental domains, then we could now relate such a system of representatives to the *cusps* of $\Gamma \setminus \mathbb{H}$; see e.g. [4, p. 363].

2 Cohomological interpretation of modular forms

From now on, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level $N \in \mathbb{N}$, and let $k \geq 2$ be a natural number.

We consider $\mathrm{Sym}^k \mathbb{C}^2$ as a left $\mathbb{C}[\Gamma]$-module, with Γ-action induced by $v \mapsto \gamma v$ for $v \in \mathbb{C}^2$ and $\gamma \in \Gamma$. For the sake of convenience, we shall identify $\mathrm{Sym}^k \mathbb{C}^2$ with the $\mathbb{C}$-vector space of homogeneous polynomials in two variables[6] X, Y of degree k:

$$\mathrm{Sym}^k \mathbb{C}^2 \cong \bigoplus_{j=0}^{k} \mathbb{C}\, X^j Y^{k-j}.$$

In this notation, the Γ-action may be described by

$$(\gamma P)(X,Y) := P\big((X,Y)\cdot\gamma\big) = P(aX + cY, bX + dY), \tag{7}$$

where $P = P(X,Y) \in \mathrm{Sym}^k \mathbb{C}^2$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Now it is a general principle[7] that we may associate to the $\mathbb{C}[\Gamma]$-module $\mathrm{Sym}^k \mathbb{C}^2$ a sheaf of $\mathbb{C}$-modules $\widetilde{\mathrm{Sym}^k \mathbb{C}^2}$ on $\Gamma \backslash \mathbb{H}$ defined as follows: If $\pi\colon \mathbb{H} \to \Gamma\backslash\mathbb{H}$ denotes the canonical projection, then $\widetilde{\mathrm{Sym}^k \mathbb{C}^2}(U)$ for open $U \subseteq \Gamma\backslash\mathbb{H}$ consists of all locally constant functions

$$\phi\colon \pi^{-1}(U) \to \mathrm{Sym}^k \mathbb{C}^2 \ \text{ satisfying } \phi(\gamma\tau) = \gamma(\phi(\tau)) \tag{8}$$

for every $\tau \in \pi^{-1}(U)$ and $\gamma \in \Gamma$.

The sheaf cohomology groups $H^\bullet(\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ may be computed by means of the *twisted de Rham complex*

$$0 \to \mathscr{C}^\infty(\mathbb{H}, \mathrm{Sym}^k \mathbb{C}^2)^\Gamma \xrightarrow{d} \Omega^1(\mathbb{H}, \mathrm{Sym}^k \mathbb{C}^2)^\Gamma \xrightarrow{d} \dots,$$

where $\Omega^i(\mathbb{H}, \mathrm{Sym}^k \mathbb{C}^2)^\Gamma$ is the $\mathbb{C}$-vector space of all smooth $\mathrm{Sym}^k \mathbb{C}^2$-valued i-forms

$$\omega = \sum_{j=0}^{k} \omega_j X^j Y^{k-j} \in \Omega^i(\mathbb{H}, \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Sym}^k \mathbb{C}^2$$

on $\mathbb{H}$ satisfying $(\gamma^*\omega)(\tau) = \gamma(\omega(\tau))$, or more explicitly,

$$\sum_{j=0}^{k} (\gamma^*\omega_j) X^j Y^{k-j} = \sum_{j=0}^{k} \omega_j (aX + cY)^j (bX + dY)^{k-j} \tag{9}$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $(\gamma^*\omega)(\tau) = \omega(\gamma\tau)$ is the pullback of ω under the map $\tau \mapsto \gamma\tau$. (Note also that $\Omega^0 := \mathscr{C}^\infty$.)

This is rather obvious if Γ is torsion-free, since then $\pi\colon \mathbb{H} \to \Gamma\backslash\mathbb{H}$ is a covering

[6] By convention, the variables X, Y correspond to the standard $\mathbb{C}$-basis $\binom{1}{0}, \binom{0}{1}$ for $\mathbb{C}^2$.

[7] More generally, if X is a topological space with a left G-action and M is an $A[G]$-module for some commutative ring A, then we have a sheaf of A-modules $\widetilde{M}$ on $G\backslash X$ defined by (8). Alternatively, one may regard $\widetilde{M}$ as the sheaf of sections of the projection $G\backslash(X \times M) \to G\backslash X$, where G acts on $X \times M$ via $g(x,m) := (gx, gm)$.

map, thus $\widetilde{\mathrm{Sym}^k \mathbb{C}^2}$ a locally constant sheaf. For the general case, we shall refer to [2, §VII.2.2].

We are mainly interested in the first cohomology group $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$, since elements of this group may be described in terms of modular forms.
Indeed, to any modular form $f \in \mathbb{M}_{k+2}(\Gamma)$ for Γ of weight $k+2$, we may associate the differential form $\delta_f \in \Omega^1(\mathbb{H}) \otimes_{\mathbb{C}} \mathrm{Sym}^k \mathbb{C}^2$ given by

$$\delta_f(\tau) := f(\tau)(\tau X + Y)^k d\tau.$$

Obviously δ_f is closed, and for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we calculate that

$$\begin{aligned}(\gamma^*\delta_f)(\tau) &= f(\tau)(c\tau+d)^{k+2}\big((a\tau+b)X + (c\tau+d)Y\big)^k (c\tau+d)^{-k}\frac{d\tau}{(c\tau+d)^2} \\ &= f(\tau)\big(\tau(aX+cY) + (bX+dY)\big)^k d\tau = \gamma(\delta_f(\tau)),\end{aligned}$$

proving that $\delta_f \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$.
Moreover, since the ω_j in (9) are only required to be smooth (and not to be holomorphic), there is the following variation of the above construction: Let us denote by $\overline{\mathbb{M}}_k(\Gamma)$ (respectively $\overline{\mathbb{S}}_k(\Gamma)$) the $\mathbb{C}$-vector space of all functions $\overline{f}\colon \mathbb{H} \to \mathbb{C}$ for $f \in \mathbb{M}_k(\Gamma)$ (respectively $f \in \mathbb{S}_k(\Gamma)$), where the bar $\bar{\cdot}$ denotes complex conjugation. To any $\overline{f} \in \overline{\mathbb{M}}_k(\Gamma)$, we may now associate

$$\delta_{\overline{f}}(\tau) := \overline{f}(\tau)(\overline{\tau}X + Y)^k d\overline{\tau} \in \Omega^1(\mathbb{H}) \otimes_{\mathbb{C}} \mathrm{Sym}^k \mathbb{C}^2 .$$

A similar calculation as above then shows that $\delta_{\overline{f}} \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$.

It may now come as a surprise to the reader that the 1-forms δ_f and $\delta_{\overline{f}}$ coming from modular forms f already exhaust all the elements from $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$: In fact, the *Eichler–Shimura isomorphism* (in the setting of de Rham cohomology) asserts that the assignment $f \mapsto \delta_f$ respecively $\overline{f} \mapsto \delta_{\overline{f}}$ gives rise to a Hecke-equivariant[8] isomorphism

$$\mathbb{M}_{k+2}(\Gamma) \oplus \overline{\mathbb{S}}_{k+2}(\Gamma) \xrightarrow{\cong} H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}). \tag{10}$$

For a proof (however, in terms of group cohomology), we refer to [14, Thm. 6.4.1, Rem. 6.4.2].
The upshot is that (10) provides a cohomological way to think about modular forms. In the next section, we will see how the notion of constant terms may be implemented into this setting.

[8] Since Hecke operators do not play a role in the construction of polylogarithmic Eisenstein classes, we shall resist from explaining in detail what Hecke-equivariance means in this context. Instead, the interested reader is referred to [11, Ch. IV] or [12, Ch. III].

3 Cohomological interpretation of constant terms

Let $f \in \mathbb{M}_{k+2}(\Gamma)$, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is (still) a congruence subgroup of level N. Then for the existence of its Fourier series expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/N},$$

it suffices to know that $f|_{k+2}\gamma = f$ for every γ in the subgroup of Γ generated by $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$. That means, if we are only interested in the constant term a_0 of f, then we are (theoretically speaking) allowed to forget the fact that $f|_{k+2}\gamma = f$ holds for *all* $\gamma \in \Gamma$.

Keeping this in mind, and identifying $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H}$ as usually via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \mapsto \frac{ai+b}{ci+d} \in \mathbb{H},$$

we observe that $\mathrm{SL}_2(\mathbb{R})$ has the subgroup

$$\mathcal{N} := \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\} \subseteq \mathrm{SL}_2(\mathbb{R})$$

isomorphic to $\mathbb{R}$, and that

$$\Gamma \cap \mathcal{N} = \left\{ \begin{pmatrix} 1 & Nz \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\}$$

is the (infinite cyclic) subgroup of Γ generated by $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Now consider the composition of maps

$$\Phi \colon \mathbb{R} \xrightarrow{\cong} \mathcal{N} \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \twoheadrightarrow \mathbb{H},$$

where the isomorphism $\mathbb{R} \xrightarrow{\cong} \mathcal{N}$ is given by $r \in \mathbb{R} \mapsto \begin{pmatrix} 1 & Nr \\ 0 & 1 \end{pmatrix} \in \mathcal{N}$.

Explicitly, this means that $\Phi(r) = Nr + i$ for $r \in \mathbb{R}$. Letting $\varphi \colon \mathbb{Z} \xrightarrow{\cong} \Gamma \cap \mathcal{N} \hookrightarrow \Gamma$ be the group homomorphism defined by

$$\varphi(z) = \begin{pmatrix} 1 & Nz \\ 0 & 1 \end{pmatrix}$$

for $z \in \mathbb{Z}$, then Φ is a *φ-map* in the sense that

$$\Phi(z + r) = \varphi(z)\Phi(r)$$

for all $z \in \mathbb{Z}$ and $r \in \mathbb{R}$. Moreover, φ induces a $\mathbb{Z}$-action on $\mathrm{Sym}^k \mathbb{C}^2$, which may (in view of (7)) be described by

$$(zP)(X, Y) := P(X, NzX + Y)$$

for $P = P(X,Y) \in \mathrm{Sym}^k \mathbb{C}^2$ and $z \in \mathbb{Z}$. In particular, the map $\mathrm{Sym}^k \mathbb{C}^2 \to \mathbb{C}$ sending $P = P(X,Y) \in \mathrm{Sym}^k \mathbb{C}^2$ to the coefficient of Y^k is a homomorphism of $\mathbb{C}[\mathbb{Z}]$-modules, where we regard $\mathbb{C}$ as $\mathbb{C}[\mathbb{Z}]$-module with trivial $\mathbb{Z}$-action.
Thus we obtain a *restriction map*

$$\begin{aligned} R^*\colon H^1(\Gamma\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) &\to H^1((\Gamma\cap\mathcal{N})\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \\ &\to H^1(\mathbb{Z}\setminus\mathbb{R}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \to H^1(\mathbb{Z}\setminus\mathbb{R}, \widetilde{\mathbb{C}}) \xrightarrow{\cong} \mathbb{C}, \end{aligned} \tag{11}$$

where $\widetilde{\mathbb{C}}$ is the sheaf on $\mathbb{Z}\setminus\mathbb{R}$ associated to the trivial $\mathbb{C}[\mathbb{Z}]$-module $\mathbb{C}$ (compare with (8)), and where the last arrow is the *de Rham isomorphism*: Explicitly, if $\omega \in \Omega^1(\mathbb{Z}\setminus\mathbb{R}, \mathbb{C})$ has Fourier series expansion

$$\omega(r) = \Big(\sum_{n\in\mathbb{Z}} b_n e^{2\pi i r n}\Big) dr,$$

then the de Rham isomorphism assigns $\omega \in H^1(\mathbb{Z}\setminus\mathbb{R}, \widetilde{\mathbb{C}}) \mapsto b_0 \in \mathbb{C}$.

For our modular form $f \in \mathbb{M}_{k+2}(\Gamma)$ from above, this means that

$$R^*(\delta_f) = a_0. \tag{12}$$

Indeed, the image of $\delta_f \in H^1(\Gamma\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ in $H^1(\mathbb{Z}\setminus\mathbb{R}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ is given by

$$f(i+Nr)\big((i+Nr)X+Y\big)^k dr = \ldots + \Big(a_0 + \sum_{n=1}^{\infty} a_n e^{-2\pi n/N}\cdot e^{2\pi i n r}\Big) dr \cdot Y^k,$$

which is sent to a_0 as desired.

However, as explained at the end of §1, a modular form $f \in \mathbb{M}_{k+2}(\Gamma)$ may have several constant terms.
In order to capture all of them (by means of cohomology), we shall construct further restriction maps. To this aim, let as assume for notational simplicity that $\Gamma = \Gamma[N]$, and let $\alpha_1, \ldots, \alpha_r \in \mathrm{SL}_2(\mathbb{Z})$ be as in (6). Then for each $1 \le j \le r$, the diffeomorphism $\mathbb{H} \to \mathbb{H}$, $\tau \mapsto \alpha_j\tau$ is a conj_{α_j}-map, where $\mathrm{conj}_{\alpha_j}\colon \Gamma \to \Gamma$ denotes the inner automorphism $\gamma \mapsto \alpha_j\gamma\alpha_j^{-1}$ (recall that $\Gamma = \Gamma[N] \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a normal subgroup). Similarly, the $\mathbb{C}$-linear map $\mathrm{Sym}^k \mathbb{C}^2 \to \mathrm{Sym}^k \mathbb{C}^2$ induced by $v \in \mathbb{C}^2 \mapsto \alpha_j v \in \mathbb{C}^2$ is a conj_{α_j}-map.
In particular, the morphism

$$\Phi_j\colon H^1(\Gamma\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \to H^1(\Gamma\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$$

induced by $\tau \mapsto \alpha_j\tau$ satisfies $\Phi_j(\delta_f) = \delta_{f|_{k+2}\alpha_j}$ for every $f \in \mathbb{M}_{k+2}(\Gamma)$. So if we define restriction maps

$$R_1^*, \ldots, R_r^*\colon H^1(\Gamma\setminus\mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \to \mathbb{C}$$

by $R_j^* := R^* \circ \Phi_j$ for every $1 \le j \le r$, then it follows that the complex numbers $R_1^*(\delta_f), \ldots, R_r^*(\delta_f)$ are precisely the constant terms of $f \in \mathbb{M}_{k+2}(\Gamma)$.
One crucial property of these restriction maps is that they preserve rationality and integrality (for a suitable interpretation of these notions). We will discuss this in §5, but beforehand, we shall introduce *Eisenstein classes*, which are our main objects of interest in this article.

4 Eisenstein classes and Eisenstein cohomology

Adopting the notation from the previous section, we see that the subspace

$$\mathbb{S}_{k+2}(\Gamma) \oplus \overline{\mathbb{S}}_{k+2}(\Gamma) \subseteq \mathbb{M}_{k+2}(\Gamma) \oplus \overline{\mathbb{S}}_{k+2}(\Gamma)$$

corresponds precisely to

$$\bigcap_{j=1}^{r} \ker(R_j^*) \subseteq H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$$

under the Eichler–Shimura isomorphism (10).

An *Eisenstein class* is now simply a cohomology class $\omega \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ such that

$$\omega \notin \bigcap_{j=1}^{r} \ker(R_j^*),$$

i.e., at least one of the constant terms $R_1^*(\omega), \ldots, R_r^*(\omega)$ of ω is non-trivial.

One could also go one step further and ask for a natural[9] complement for $\bigcap_{j=1}^{r} \ker(R_j^*)$ in $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$. It is clear that such a complement (in the sense of $\mathbb{C}$-vector spaces) may be spanned by suitable Eisenstein classes, but there are many different ways of doing this.
However, there is a natural complement $\mathbb{E}\mathrm{is}_{k+2}(\Gamma) \subseteq \mathbb{M}_{k+2}(\Gamma)$ for $\mathbb{S}_{k+2}(\Gamma)$, consisting of all those $f \in \mathbb{M}_{k+2}(\Gamma)$ satisfying $\langle f, g \rangle = 0$ for every $g \in \mathbb{S}_{k+2}(\Gamma)$, where $\langle \cdot, \cdot \rangle$ denotes the *Petersson scalar product* (see [12, §III.5] or [14, §6.1] for a definition). Then $\mathbb{E}\mathrm{is}_{k+2}(\Gamma)$ admits a basis consisting of eigenfunctions for all the Hecke operators (cf. [12, p. 174]), so that $\mathbb{E}\mathrm{is}_{k+2}(\Gamma)$ is even a module under the Hecke algebra.
Anyway, we denote the image of $\mathbb{E}\mathrm{is}_{k+2}(\Gamma)$ under the Eichler–Shimura isomorphism by

$$H^1_{\mathrm{Eis}}(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \subseteq H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$$

and call it the *Eisenstein cohomology* (of degree 1) for the pair $(\Gamma, \mathrm{Sym}^k \mathbb{C}^2)$. By construction, we may now write

$$H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \cong \bigcap_{j=1}^{r} \ker(R_j^*) \oplus H^1_{\mathrm{Eis}}(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}),$$

which is actually a direct sum of Hecke modules.
In this sense, an Eisenstein class $\omega \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ is not necessarily contained in the Eisenstein cohomology, but the projection of ω onto $H^1_{\mathrm{Eis}}(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ gives a non-trivial element of the latter.
For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then it follows from [12, Prop. 40, p. 163] that $\mathbb{E}\mathrm{is}_{k+2}(\Gamma) = \mathbb{C} \cdot G_{k+2}$, where G_{k+2} was introduced in (2), hence that

$$H^1_{\mathrm{Eis}}(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) = \mathbb{C} \cdot G_{k+2}.$$

[9] Basically, it is part of the question to figure out what "natural" could mean here.

To conclude this section, we shall illustrate the "classical way" of constructing Eisenstein classes in the special case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and even $k \geq 2$. A different approach to the construction of such classes (via the so-called topological polylogarithm) will be presented in §§6–7.

The idea is to start with a class in

$$H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) \cong H^1((\Gamma \cap \mathcal{N}) \setminus \mathcal{N}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2});$$

for example let us take $((r+i)X+Y)^k dr \in \Omega^1(\mathbb{Z} \setminus \mathbb{R}, \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Sym}^k \mathbb{C}^2$. This is the pullback of $\omega := (\tau X+Y)^k d\tau$, which defines a class in $H^1((\Gamma \cap \mathcal{N})^{\pm} \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ but not in $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$, where

$$(\Gamma \cap \mathcal{N})^{\pm} := \left\{ \pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\} \subseteq \Gamma.$$

To solve this problem, we construct

$$\omega'(\tau) := \sum_{\beta \in \mathcal{R}} \beta^{-1}(\omega(\beta\tau)),$$

where $\mathcal{R}$ is a system of representatives for $(\Gamma \cap \mathcal{N})^{\pm} \setminus \Gamma$. Note that the elements of $\mathcal{R}$ are only unique up to $(\Gamma \cap \mathcal{N})^{\pm}$-multiples from the left. However, given $\gamma \in (\Gamma \cap \mathcal{N})^{\pm}$, we find that

$$(\gamma\beta)^{-1}(\omega(\gamma\beta\tau)) = (\gamma\beta)^{-1}\gamma\omega(\beta\tau) = \beta^{-1}\omega(\beta\gamma),$$

so ω' does not depend on the choice of $\mathcal{R}$. Moreover, for any $\alpha \in \Gamma$, also $\mathcal{R} \cdot \alpha$ is a system of representatives for $(\Gamma \cap \mathcal{N})^{\pm} \setminus \Gamma$. Thus

$$\omega'(\alpha\tau) = \sum_{\beta \in \mathcal{R}} \beta^{-1}(\omega(\beta\alpha\tau)) = \sum_{\eta \in \mathcal{R}\cdot\alpha} \alpha\eta^{-1}(\omega(\eta\tau)) = \alpha\omega'(\tau),$$

proving that $\omega' \in H^1(\Gamma \setminus \mathbb{H}, \mathrm{Sym}^k \mathbb{C}^2)$ *provided that* ω' is a meaningful object (e.g., in the sense of convergence). However, a direct computation reveals that

$$\omega'(\tau) = (2\zeta(k+2))^{-1} G_{k+2}(\tau)(\tau X + Y)^k d\tau = (2\zeta(k+2))^{-1} \delta_{G_{k+2}},$$

which is certainly well-defined. However, we see that the success of this procedure crucially relies on the initial choice of ω.

Finally, it is important to remark that the theory of Eisenstein cohomology (in its full generality) is by far more sophisticated than what we presented here. The interested reader is encouraged to consult [5], [6] or [13].

5 The key idea

Now we have spent quite some time on interpreting modular forms and their constant terms cohomologically. But what are the advantages of this point of view?

Let $A \subseteq \mathbb{C}$ be any (noetherian) subring. Then we may regard $\operatorname{Sym}^k A^2$ as an $A[\Gamma]$-module, with Γ-action given by (7). In particular, we obtain a sheaf $\widetilde{\operatorname{Sym}^k A^2}$ of A-modules on $\Gamma \setminus \mathbb{H}$ (see (8)), and morphisms of A-modules

$$\begin{aligned} &H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k A^2}) \to H^1((\Gamma \cap \mathcal{N}) \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k A^2}) \\ \to &H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{\operatorname{Sym}^k A^2}) \to H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{A}) \end{aligned} \tag{13}$$

as in (11). Recall that in the case of $A = \mathbb{C}$, we obtained the restriction map $R^* =: R^*_{\mathbb{C}}$ by post-composing (13) with the de Rham isomorphism

$$H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{\mathbb{C}}) \xrightarrow{\cong} \mathbb{C}\,.$$

Now it is a non-trivial fact (see e.g. [8, (3.3.15),(3.3.16)]) that there exists a morphism of A-modules $H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{A}) \to A$ so that the following diagram commutes:

$$\begin{array}{ccc} H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{A}) & \longrightarrow & A \\ \downarrow & & \downarrow \\ H^1(\mathbb{Z} \setminus \mathbb{R}, \widetilde{\mathbb{C}}) & \xrightarrow{\text{de Rham} \cong} & \mathbb{C} \end{array}.$$

Combining the latter with (13), we obtain a restriction map

$$R^*_A \colon H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k A^2}) \to A$$

which fits into the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k A^2}) & \xrightarrow{R^*_A} & A \\ \downarrow & & \downarrow \\ H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k \mathbb{C}^2}) & \xrightarrow{R^*_{\mathbb{C}}} & \mathbb{C} \end{array}. \tag{14}$$

As a consequence (and this is the key idea):

If $\omega \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k \mathbb{C}^2})$ is the image of a cohomology class from $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\operatorname{Sym}^k A^2})$, then $R^*_{\mathbb{C}}(\omega) \in A$.

Of course, this is of interest only for Eisenstein classes ω. In view of the previous sections (in particular, recall (4) and (12)), we might hope that $R^*_{\mathbb{C}}(\omega)$ is (a suitable multiple of) a special value of Riemann's zeta function. Then the fact that $R^*_{\mathbb{C}}(\omega) \in A$ provides some information on the arithmetic nature of the zeta value.

If we content ourselves with working over $A = \mathbb{Q}$, then the commutative square (14) is perfectly fine for our purpose (of constructing polylogarithmic Eisenstein classes). However, in order to get integrality results (which refers to choices like $A = \mathbb{Z}_{(p)}$ for a prime p or even $A = \mathbb{Z}[M^{-1}]$ for some $M \in \mathbb{N}$), we need to modify the above construction slightly. The necessity of these modifications will become apparent during our construction of polylogarithmic Eisenstein classes in §§6–7.

On the one hand, we work with $\mathrm{TSym}^k A^2 \subseteq (A^2)^{\otimes k}$ instead of $\mathrm{Sym}^k A^2$. Given a free A-module M of finite rank, we recall that an element $\alpha \in M^{\otimes k}$ is contained in $\mathrm{TSym}^k M$ precisely if $\sigma(\alpha) = \alpha$ for every permutation $\sigma \in \mathfrak{S}_k$, which we identify with the unique A-linear map $M^{\otimes k} \to M^{\otimes k}$ sending $v_1 \otimes \cdots \otimes v_k$ to $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. Letting $l_1, \ldots, l_n$ be an A-basis for M, we have a homomorphism of A-modules

$$\begin{aligned} \mathrm{Sym}^k M &\to \mathrm{TSym}^k M \\ l_1^{\otimes k_1} \otimes \cdots \otimes l_n^{\otimes k_n} &\mapsto k_1! \cdots k_n! \cdot l_1^{\otimes k_1} \cdots l_n^{\otimes k_n}, \end{aligned} \tag{15}$$

where $k = k_1 + \ldots + k_n$, and where the product on the right hand side is the *shuffle product*[10]. Piecing these together, we obtain a homomorphism of A-algebras

$$\prod_{k \geq 0} \mathrm{Sym}^k M \to \prod_{k \geq 0} \mathrm{TSym}^k M, \tag{16}$$

which is even an isomorphism if A is a $\mathbb{Q}$-algebra.
Moreover, for any subgroup Γ of A-module automorphisms of M, we see that $\mathrm{TSym}^k M$ inherits from $M^{\otimes k}$ the structure of a Γ-module, turning (15) and (16) into homomorphisms of $A[\Gamma]$-modules.
Similarly as for $\mathrm{Sym}^k A^2$, we now obtain a homomorphism of $A[\mathbb{Z}]$-modules $\mathrm{TSym}^k A^2 \to A$ as follows: Let e_1, e_2 be the standard A-basis for A^2, then the elements $e_1^{\otimes k_1} \cdot e_2^{\otimes k_2}$ with $k_1 + k_2 = k$ form an A-basis for $\mathrm{TSym}^k A^2$. Given this, we may let $\mathrm{TSym}^k A^2 \to A$ send any element of $\mathrm{TSym}^k A^2$ to the coefficient of $e_2^{\otimes k}$ with respect to this basis.

On the other hand, instead of associating to $\mathrm{TSym}^k A^2$ the sheaf of A-modules $\widetilde{\mathrm{TSym}^k} A^2$ on $\Gamma \backslash \mathbb{H}$ defined by (8), we consider the constant sheaf $\underline{\mathrm{TSym}^k A^2}$ regarded as a Γ-*equivariant sheaf* of A-modules on $\mathbb{H}$. This means that we have a family of isomorphisms

$$\underline{\mathrm{TSym}^k A^2} \xrightarrow{\cong} \gamma^{-1} \underline{\mathrm{TSym}^k A^2}$$

indexed by $\gamma \in \Gamma$ satisfying a certain cocycle condition. Explicitly, regarding elements of $\underline{\mathrm{TSym}^k A^2}(U)$ for open $U \subseteq \mathbb{H}$ as locally constant functions $U \to \mathrm{TSym}^k A^2$, the above isomorphism sends $\phi \colon U \to \mathrm{TSym}^k A^2$ to $\gamma\phi \colon \gamma U \to \mathrm{TSym}^k A^2$ defined by

$$(\gamma\phi)(\tau) = \gamma(\phi(\gamma^{-1}\tau))$$

for $\tau \in U$. The point is that we may now consider the *equivariant cohomology*[11]

[10] Given $k, l \geq 0$, the shuffle product

$$\mathrm{TSym}^k M \times \mathrm{TSym}^l M \to \mathrm{TSym}^{k+l} M$$

sends $(\alpha, \beta) \in \mathrm{TSym}^k M \times \mathrm{TSym}^l M$ to $\sum_{\sigma \in \mathcal{S}_{k,l}} \sigma(\alpha \otimes \beta)$, where $\mathcal{S}_{k,l}$ is any set of representatives for $\mathfrak{S}_{k+l}/(\mathfrak{S}_k \times \mathfrak{S}_l)$.

[11] Given a topological space X equipped with a left G-action and a commutative ring A, the equivariant cohomology groups $H^\bullet(X, G; *)$ are the derived functors of $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})^G$, which is a functor from G-equivariant sheaves of A-modules on X to A-modules.

group $H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2})$. Note that one always has a homomorphism of A-modules[12]

$$H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{TSym}^k A^2}) \to H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2}),$$

which is an isomorphism if Γ is torsion-free or if A is a $\mathbb{Q}$-algebra. In particular, one has morphisms

$$H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2}) \to H^1(\mathbb{H}, \Gamma \cap \mathcal{N}; \underline{\mathrm{TSym}^k A^2}) \xleftarrow{\cong} H^1((\Gamma \cap \mathcal{N}) \setminus \mathbb{H}, \widetilde{\mathrm{TSym}^k A^2}),$$

where we used that $\Gamma \cap \mathcal{N} \cong \mathbb{Z}$ is torsion-free. In this way, we obtain a modified restriction map

$$\widehat{R}^*_A \colon H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2}) \to A$$

such that

$$\begin{array}{ccccc} H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2}) & \xrightarrow{\quad \widehat{R}^*_A \quad} & A & & \\ \downarrow & & & \searrow^{\cdot (k!)^{-1}} & \\ H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k \mathbb{C}^2}) & & & & \mathbb{C} \\ \cong \uparrow & & & \nearrow_{R^*_{\mathbb{C}}} & \\ H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{TSym}^k \mathbb{C}^2}) & \xleftarrow[\cong]{} & H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2}) & & \end{array} \tag{17}$$

commutes. Note that $A \to \mathbb{C}$ is given by multiplication by $(k!)^{-1}$ because $e_2^{\otimes k} \mapsto k! e_2^{\otimes k}$ under the isomorphism $\mathrm{Sym}^k \mathbb{C}^2 \xrightarrow{\cong} \mathrm{TSym}^k \mathbb{C}^2$ described in (15). As a consequence:

If $\omega \in H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k \mathbb{C}^2})$ *comes from a class in* $H^1(\mathbb{H}, \Gamma; \underline{\mathrm{TSym}^k A^2})$*, then* $k! \cdot R^*_{\mathbb{C}}(\omega) \in A$.

6 The topological polylogarithm

In this section, we fix a natural number $n \in \mathbb{N}$ and a commutative (noetherian) ring A. We consider the completed group algebra

$$R_A := A[|\, \mathbb{Z}^n \,|] := \varprojlim_{k \geq 1} A[\mathbb{Z}^n]/J^k,$$

where $J := \ker(A[\mathbb{Z}^n] \xrightarrow{\text{aug}} A)$ is the augmentation ideal. Note that R_A is (non-canonically) isomorphic to the formal power series ring $A[|X_1, \dots, X_n|]$, and that one has a homomorphism of $A[\mathbb{Z}^n]$-algebras

$$R_A \to \prod_{k \geq 0} \mathrm{TSym}^k A^2, \tag{18}$$

where $z \in \mathbb{Z}^n$ acts on $\prod_{k \geq 0} \mathrm{TSym}^k A^2$ by multiplication by $\sum_{k \geq 0} z^{\otimes k}$, which is even an isomorphism if A is a $\mathbb{Q}$-algebra. In this case, combined with (16), we obtain an isomorphism of $A[\mathbb{Z}^n]$-algebras

$$R_A \xrightarrow{\cong} \prod_{k \geq 0} \mathrm{Sym}^k A^2, \tag{19}$$

[12] being *edge morphism* of a certain spectral sequence

where $z \in \mathbb{Z}^n$ acts on $\prod_{k\geq 0} \operatorname{Sym}^k A^2$ by multiplication by $\sum_{k\geq 0} z^{\otimes k}/k!$. For the details, we refer to [1, §3.1].
Furthermore, since R_A is an $A[\mathbb{Z}^n]$-module, it gives rise to a locally constant sheaf of A-modules $\mathscr{L}\mathrm{og}_A$ on the torus

$$T := \mathbb{Z}^n \setminus \mathbb{R}^n$$

defined by (8), which we call the *logarithm sheaf* on T.

However, R_A even has the structure of an $A[\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})]$-module, where the group operation of $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ is given by

$$(z_1, \gamma_1) \cdot (z_2, \gamma_2) := (z_1 + \gamma_1 z_2, \gamma_1\gamma_2),$$

and where $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ acts on R_A via

$$(z, \gamma)r := (z) \cdot \varphi_\gamma(r),$$

for $\varphi_\gamma \colon R_A \to R_A$ being the A-algebra homomorphism induced by $z \in \mathbb{Z}^2 \mapsto \gamma z \in \mathbb{Z}^2$. This turns $\mathscr{L}\mathrm{og}_A$ into a $\mathrm{GL}_n(\mathbb{Z})$-*equivariant sheaf*, where for any open $U \subseteq T$ and $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ the isomorphism

$$\rho_\gamma(U) \colon \mathscr{L}\mathrm{og}_A(U) \xrightarrow{\cong} \mathscr{L}\mathrm{og}_A(\gamma U)$$

sends $\phi \colon \pi^{-1}(U) \to R_A$ to $\gamma\phi \colon \pi^{-1}(\gamma U) = \gamma\pi^{-1}(U) \to R_A$ defined by

$$(\gamma\phi)(x) := \varphi_\gamma(\phi(\gamma^{-1}x))$$

for $x \in \pi^{-1}(\gamma U)$, where $\pi \colon \mathbb{R}^n \to T$ is the canonical projection.
Moreover, one checks that the morphisms in (18) and (19) respect this structure.

Let now $M \in \mathbb{N}$ be a natural number such that $M \in A^\times$. Further, let λ be the orientation sheaf on T, which we may describe as the Γ-equivariant sheaf of $\mathbb{Z}$-modules on T associated to the $\mathbb{Z}[\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})]$-module $\mathbb{Z}$, where $(z, \gamma) \in \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{Z}$ by multiplication by $\det(\gamma)^{-1}$. Furthermore, denote by

$$T[M] := \mathbb{Z}^n \setminus M^{-1}\mathbb{Z}^n \subseteq T$$

the M-torsion points of T, write $D := T[M] \setminus \{0\}$, and let

$$\Gamma := \{\gamma \in \mathrm{GL}_n(\mathbb{Z}) \colon \gamma D = D\}.$$

Then one computes that

$$H^{n-1}(T \smallsetminus D, \Gamma; \mathscr{L}\mathrm{og}_A \otimes \lambda) \cong \ker\Big(\bigoplus_{d\in D} R_A \to A\Big)^\Gamma, \tag{20}$$

see [1, Cor. 3.28], where elements of the right hand side may be regarded as functions $\alpha \colon D \to R_A$ satisfying

$$\alpha \circ \gamma = \varphi_\gamma \circ \alpha \quad \text{and} \quad \sum_{d\in D} \operatorname{aug}(\alpha(d)) = 0$$

for every $\gamma \in \Gamma$. Given any

$$\alpha \in \ker\big(\bigoplus_{d\in D} A \to A\big)^{\Gamma} \subseteq \ker\big(\bigoplus_{d\in D} R_A \to A\big)^{\Gamma},$$

the *topological polylogarithm* pol_α associated to α is the unique equivariant cohomology class in $H^{n-1}(T \smallsetminus D, \Gamma; \mathscr{L}\mathrm{og}_A \otimes \lambda)$ corresponding to α under (20).

For readers who do not feel comfortable with equivariant cohomology, we shall provide a different description of $H^{n-1}(T \smallsetminus D, \Gamma; \mathscr{L}\mathrm{og} \otimes \lambda)$ in the special case where A is a $\mathbb{Q}$-algebra: Let $\mathscr{P}_n$ be the space of real symmetric positive definite $n \times n$-matrices, on which $\gamma \in \Gamma$ acts via $B \in \mathscr{P}_n \mapsto \gamma B \gamma^t \in \mathscr{P}_n$. Since $\mathscr{P}_n$ is contractible, we then have

$$\begin{aligned} H^{n-1}(T \smallsetminus D, \Gamma; \mathscr{L}\mathrm{og}_A \otimes \lambda) &\cong H^{n-1}((T \smallsetminus D) \times \mathscr{P}_n, \Gamma; \mathscr{L}\mathrm{og}_A \otimes \lambda) \\ &\cong H^{n-1}(\Gamma \backslash ((T \smallsetminus D) \times \mathscr{P}_n), \mathscr{L}\mathrm{og}_A \otimes \lambda), \end{aligned}$$

where we do not (notationally) distinguish between the logarithm sheaves on T and $T \times \mathscr{P}_n$. However, using equivariant cohomology we see that the topological polylogarithm is intrinsic to the torus T, i.e., that $\mathscr{P}_n$ is just a dummy space.

Anyway, the zero section $0\colon \mathscr{P}_n \to (T \smallsetminus D) \times \mathscr{P}_n, B \mapsto (0, B)$ gives rise to a commutative diagram

$$\begin{array}{ccc} H^{n-1}((T \smallsetminus D) \times \mathscr{P}_n, \Gamma; \mathscr{L}\mathrm{og}_A \otimes \lambda) & \longrightarrow & H^{n-1}(\mathscr{P}_n, \Gamma; \underline{R_A} \otimes \lambda) \\ \downarrow & & \downarrow \\ H^{n-1}((T \smallsetminus D) \times \mathscr{P}_n, \Gamma; \mathscr{L}\mathrm{og}_{\mathbb{C}} \otimes \lambda) & \longrightarrow & H^{n-1}(\mathscr{P}_n, \Gamma; \underline{R_{\mathbb{C}}} \otimes \lambda) \end{array},$$

where $R_{\mathbb{C}} \cong \prod_{k\geq 0} \mathrm{Sym}^k \mathbb{C}^2$ by (19). As discussed in §2, the cohomology group

$$H^{n-1}(\mathscr{P}_n, \Gamma; \underline{R_{\mathbb{C}}} \otimes \lambda) \cong H^{n-1}(\Gamma \backslash \mathscr{P}_n, \widetilde{R_{\mathbb{C}}} \otimes \lambda)$$

may be computed by means of a twisted de Rham complex, i.e., its elements may be represented by closed differential forms $\omega \in \Omega^1(\mathscr{P}_n, \mathbb{C}) \otimes_{\mathbb{C}} R_{\mathbb{C}} \otimes \lambda$ satisfying

$$\gamma^* \omega = \det(\gamma)^{-1} \varphi_\gamma(\omega)$$

for every $\gamma \in \Gamma$. A crucial property of the topological polylogarithm is that we may explicitly write down a differential form $\mathscr{G}_\alpha$ representing the image of $0^* \mathrm{pol}_\alpha \in H^{n-1}(\mathscr{P}_n, \Gamma; \underline{R_A} \otimes \lambda)$ in $H^{n-1}(\Gamma \backslash \mathscr{P}_n, \widetilde{R_{\mathbb{C}}} \otimes \lambda)$; compare[13] with [1, Lemma 4.12]. To this aim, choose a $\mathbb{Z}$-basis $l_1, \ldots, l_n$ for $\mathbb{Z}^n$, and denote by $\mu_1, \ldots, \mu_n$ the corresponding dual basis. Then

$$\mathscr{G}_\alpha(B) = \sum_{\mu \in \mathbb{Z}^n \setminus \{0\}} E_\mu(B) \Big(\sum_{t \in T[M]} \alpha(t) e^{2\pi i \mu(t)} \Big),$$

where $\mu(t) := \mu \cdot t$ (regarding μ as a row vector and t as a column vector), and

$$E_\mu(B) = \sum_{k\geq 0} (-1)^{k+n} \frac{(k+n-1)!}{k!} \frac{(B\mu^t)^{\otimes k}}{(2\pi i \cdot \mu B \mu^t)^{k+n}} \cdot \sum_{j=1}^{n} (-1)^j \mu_j B \mu^t \cdot$$

[13]However, be aware that we use left actions on topological spaces, whereas [1] works with right actions.

$$\cdot \bigwedge_{\substack{1\le h\le n,\\ h\neq j}} d(\mu_h B\mu^t) \otimes (l_1 \wedge \ldots \wedge l_n).$$

This formula enables to carry out explicit calculations (in terms of de Rham cohomology), as will be done in the forthcoming section.

7 Polylogarithmic Eisenstein classes for SL_2

We shall now use the topological polylogarithm to construct possible candidates for Eisenstein classes in $H^1(\Gamma \setminus \mathbb{H}, \widetilde{\mathrm{Sym}^k}\, \mathbb{C}^2)$. To this aim, let $n = 2$, fix $M \in \mathbb{N}$ and let

$$\alpha\colon T[M] = \mathbb{Z}^2 \setminus M^{-1}\,\mathbb{Z}^2 \to \mathbb{Z}[M^{-1}]$$

be any function satisfying $\alpha(0) = \sum_{t\in T[M]} \alpha(t) = 0$. Further, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of all γ satisfying $\alpha \circ \gamma = \gamma$, and let $N \in \mathbb{N}$ be any natural number[14] such that $\Gamma[N] \subseteq \Gamma$. Moreover, since all matrices in Γ have determinant 1, the orientation sheaf λ has trivial Γ-action, hence may be *trivialized* (which in the current context just amounts to "omitting" λ).

Let $\mathbb{Z}[M^{-1}] \subseteq A \subseteq \mathbb{C}$ be any (noetherian) subring. Taking the pullback of $0^* \operatorname{pol}_\alpha \in H^1(\mathscr{P}_2, \Gamma; R_A)$ under the Γ-equivariant map

$$\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}) \to \mathscr{P}_2$$

given by $g \in \mathrm{SL}_2(\mathbb{R}) \mapsto gg^t \in \mathscr{P}_2$, followed by the morphism of Γ-equivariant sheaves on $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$ induced by

$$R_A \xrightarrow{(18)} \prod_{l\ge 0} \mathrm{TSym}^l A^2 \xrightarrow{\mathrm{proj}_k} \mathrm{TSym}^k A^2,$$

we obtain a cohomology class

$$\widetilde{\operatorname{pol}}_\alpha^k \in H^1(\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}), \Gamma; \underline{\mathrm{TSym}^k A^2}) \tag{21}$$

whose image in

$$H^1(\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}), \Gamma; \underline{\mathrm{TSym}^k \mathbb{C}^2}) \cong H^1\big(\Gamma \setminus \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}), \widetilde{\mathrm{Sym}^k}\, \mathbb{C}^2\big)$$

may be represented by a differential form $\widetilde{\mathscr{G}}_\alpha^k$ on $\mathrm{SL}_2(\mathbb{R})$ given by

$$\widetilde{\mathscr{G}}_\alpha^k(g) = \sum_{\mu\in\mathbb{Z}^2\setminus\{0\}} E_\mu^k(g)\big(\sum_{t\in T[M]} \alpha(t) e^{2\pi i\mu(t)}\big),$$

where

$$E_\mu^k(g) = \frac{(-1)^{k+1}(k+1)}{(2\pi i)^{k+2}} \cdot \frac{(gg^t\mu^t)^{\otimes k}}{(\mu gg^t\mu^t)^{k+2}} \cdot \big(\mu_1 gg^t\mu^t \cdot d(\mu_2 gg^t\mu^t) - \mu_2 gg^t\mu^t \cdot d(\mu_1 gg^t\mu^t)\big)$$

[14] We may always let $N = M$, since $\Gamma[M]$ acts trivially on $T[M]$, but sometimes it may be desirable to choose N smaller.

for $\mu_1 = (1,0), \mu_2 = (0,1) \in \mathbb{Z}^2$.

Let us now compute the image of $\widetilde{\mathscr{G}}^k_\alpha$ under the restriction map (11). To this aim, we first comute the pullback of $\widetilde{\mathscr{G}}^k_\alpha$ under the $\mathbb{Z}$-equivariant map $\mathbb{R} \to \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$ given by

$$r \mapsto \begin{pmatrix} 1 & Nr \\ 0 & 1 \end{pmatrix},$$

where we recall that $\mathbb{Z}$ acts on $\mathbb{R}$ via translation $r \mapsto z + r$ and on $\mathrm{SL}_2(\mathbb{R})$ via the group homomorphism

$$z \in \mathbb{Z} \mapsto \begin{pmatrix} 1 & Nz \\ 0 & 1 \end{pmatrix} \in \Gamma[N].$$

To this aim, write

$$g = \begin{pmatrix} 1 & Nr \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ and } \mu = (a,b) \in \mathbb{Z}^2,$$

then

$$gg^t = \begin{pmatrix} 1 & Nr \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ Nr & 1 \end{pmatrix} = \begin{pmatrix} 1+N^2r^2 & Nr \\ Nr & 1 \end{pmatrix},$$

$$gg^t\mu^t = \begin{pmatrix} a(N^2r^2+1)+bNr \\ aNr+b \end{pmatrix},$$

$$\begin{aligned} \mu gg^t\mu^t &= (a,b)\begin{pmatrix} 1+N^2r^2 & Nr \\ Nr & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (a,b)\begin{pmatrix} a(N^2r^2+1)+bNr \\ aNr+b \end{pmatrix} \\ &= a^2(N^2r^2+1) + 2abNr + b^2, \end{aligned}$$

$$\mu_1 gg^t\mu^t = a(N^2r^2+1)+bNr, \;\; \mu_2 gg^t\mu^t = aNr+b,$$

$$d(\mu_1 gg^t\mu^t) = (2aNr+b)N\cdot dr, \;\; d(\mu_2 gg^t\mu^t) = aN\cdot dr.$$

Furthermore, regarding $(gg^t\mu^t)^{\otimes k} \in \mathrm{Sym}^k\,\mathbb{C}^2$ as a homogeneous polynomial in X, Y of degree k, then the coefficient of Y^k is given by $(aNr+b)^k$.

Thus $R^*(\widetilde{\mathscr{G}}^k_\alpha)$ is precisely the constant term of

$$r \mapsto \sum_{\mu=(a,b)\neq(0,0)} E^k_\mu(r)\Big(\sum_{t=(t_1,t_2)\in T[M]} \alpha(t)e^{2\pi i(at_1+bt_2)}\Big), \tag{22}$$

where

$$\begin{aligned} E^k_\mu(r) = \frac{(-1)^{k+1}(k+1)}{(2\pi i)^{k+2}} &\cdot \frac{(aNr+b)^k}{(a^2(N^2r^2+1)+2abNr+b^2)^{k+2}} \cdot \\ &\cdot\big(a^2(N^2r^2+1)+abNr-(aNr+b)(2aNr+b)\big)N\cdot dr \\ = \frac{(-1)^k(k+1)}{(2\pi i)^{k+2}} &\cdot \frac{(aNr+b)^k}{((aNr+b)^2+a^2)^{k+2}} \cdot \big((aNr+b)^2-a^2\big)N\cdot dr. \end{aligned}$$

To get an explicit expression, we compute

$$\int_0^1 \frac{(aNx+b)^k}{((aNx+b)^2+a^2)^{k+2}} \cdot \big((aNx+b)^2-a^2\big)N dx$$

$$= \frac{1}{a}\int_b^{Na+b} \frac{y^k(y^2-a^2)}{(y^2+a^2)^{k+2}} dy$$

$$= \frac{1}{(k+1)a}\Big(\frac{b^{k+1}}{(a^2+b^2)^{k+1}} - \frac{(Na+b)^{k+1}}{(a^2+(Na+b)^2)^{k+1}}\Big)$$

provided that $a \neq 0$. On the other hand, if $a = 0$, then the above integral reduces to

$$\int_0^1 \frac{b^k}{b^{2(k+2)}} b^2 N dx = \frac{N}{b^{k+2}}.$$

Now we decompose the summation in (22) as

$$\sum_{(a,b)\neq(0,0)} = \sum_{a=0,b\neq 0} + \sum_{a\neq 0}\sum_{b}$$

and claim that the second term does not contribute to the constant term. Indeed, for fixed $a \neq 0$, the contribution of the latter is a suitable multiple of

$$\sum_b \Big(\frac{b^{k+1}}{(a^2+b^2)^{k+1}} - \frac{(Na+b)^{k+1}}{(a^2+(Na+b)^2)^{k+1}}\Big)\cdot\Big(\sum_{t=(t_1,t_2)\in\mathbb{Z}^2\setminus M^{-1}\mathbb{Z}^2} \alpha(t)e^{2\pi i(a,b)\binom{t_1}{t_2}}\Big).$$

However, letting $\gamma = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$, then

$$\sum_{t=(t_1,t_2)\in T[M]} \alpha(t)e^{2\pi i(a,b)\binom{t_1}{t_2}} = \sum_{t=(t_1,t_2)\in T[M]} \alpha(\gamma^{-1}t)e^{2\pi i(a,b)\binom{t_1}{t_2}}$$

$$= \sum_{t=(t_1,t_2)\in T[M]} \alpha(t)e^{2\pi i(a,b)\gamma\binom{t_1}{t_2}} = \sum_{t=(t_1,t_2)\in T[M]} \alpha(t)e^{2\pi i(a,Na+b)\binom{t_1}{t_2}},$$

which suggests that the above series $\sum_b$ vanishes for every $a \neq 0$.
We thus arrive at

$$R^*(\widetilde{\mathscr{G}}^k_\alpha) = \frac{(-1)^{k+1}(k+1)}{(2\pi i)^{k+2}} N \sum_{b\neq 0} \frac{1}{b^{k+2}}\Big(\sum_{t=(t_1,t_2)\in T[M]} \alpha(t)e^{2\pi i b t_2}\Big). \qquad (23)$$

8 Special values of the Riemann zeta function

Let now $k \in \mathbb{N}$ be even, and consider the following choice for α: We let $M = N = q > 2$ be a prime number, and define

$$\alpha\colon \Big\{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\Big\}^2 \cong T[q] \to \mathbb{Z}[q^{-1}]$$

by

$$\alpha\Big(\frac{t_1}{q}, \frac{t_2}{q}\Big) = \begin{cases} 0 & \text{if } t_1 = t_2 = 0, \\ q & \text{if } t_1 \neq 0, t_2 = 0, \\ -1 & \text{if } t_1, t_2 \neq 0. \end{cases}$$

Since $\sum_{x\in(\mathbb{Z}/q\mathbb{Z})^\times} e^{2\pi i/q\cdot bx}$ vanishes unless $q \mid b$, we see that

$$\sum_{t=(t_1,t_2)\in T[q]} \alpha(t)e^{2\pi i b t_2} = \begin{cases} q(q-1) & \text{if } q \nmid b, \\ 0 & \text{if } q \mid b. \end{cases}.$$

Applying (23), we thus get

$$\begin{aligned} R^*(\widetilde{\mathscr{G}}^k_\alpha) &= \frac{-(k+1)}{(2\pi i)^{k+2}} 2q^2(q-1) \sum_{b>0, q\nmid b} \frac{1}{b^{k+2}} \\ &= \frac{-(k+1)}{(2\pi i)^{k+2}} 2q^2(q-1)\Big(\zeta(k+2) - \sum_{b=1}^{\infty} \frac{1}{(qb)^{k+2}}\Big) \\ &= \frac{-(k+1)}{(2\pi i)^{k+2}} 2q^2(q-1)(1-q^{-(k+2)})\zeta(k+2). \end{aligned}$$

Using (5), this simplifies to

$$R^*(\widetilde{\mathscr{G}}^k_\alpha) = -\frac{1}{k!} q^2(q-1)(1-q^{-(k+2)})\zeta(-1-k). \tag{24}$$

Now recall from (21) that $\widetilde{\mathscr{G}}^k_\alpha$ comes from the cohomology class

$$\widetilde{\mathrm{pol}}_\alpha \in H^1(\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}), \Gamma; R_A),$$

where $\mathbb{Z}[q^{-1}] \subseteq A \subseteq \mathbb{C}$ is any (noetherian) subring. If we let $A = \mathbb{Q}$, then it follows immediately from the commutative square (14) that $\zeta(-1-k) \in \mathbb{Q}$. However, we can say by far more about these zeta values:
Indeed, let p be any prime, and choose q in such a way that it is a primitive root modulo p (which is possible by Dirichlet's prime number theorem). So $p \nmid q-1$ unless $p = 2$, and if $p \mid q^{k+2}-1$, then this implies $p-1 = \mathrm{ord}_p(q) \mid k+2$, where $\mathrm{ord}_p(q)$ denotes the order of $\overline{q}$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. That means, if we further assume that $p-1 \nmid k+2$, then $q^2(q-1)(1-q^{-(k+2)}) \in \mathbb{Z}_{(p)}^\times$ and it follows from the commutative diagram (17) that

$$\zeta(-1-k) \in \mathbb{Z}_{(p)}\,. \tag{25}$$

As a consequence, $p-1 \mid k+2$ is a necessary condition for p to appear as a prime factor in the denominator of $\zeta(-1-k)$.
But even if this is the case (i.e., $p-1 \mid k+2$), we can still say that

$$(q-1)(q^{k+2}-1)\zeta(-1-k) \in \mathbb{Z}_{(p)}\,.$$

One can now try to minimize

$$\nu_p\big((q-1)(q^{k+2}-1)\big),$$

where ν_p is the p-adic valuation, which then translates into an upper bound for the power of p appearing in the denominator of $\zeta(-1-k)$.
For instance, if $p > 2$, then we may let q be a primitive root modulo $p^{\nu_p(k+2)+2}$, which yields

$$\nu_p\big((q-1)(q^{k+2}-1)\big) = \nu_p(k+2) + 1. \tag{26}$$

In the case $p = 2$, we face the problem that $(\mathbb{Z}/2^l\,\mathbb{Z})^\times$ fails to be cyclic for $l \geq 3$. Nevertheless, choosing q in such a way that $q \equiv -5 \pmod{2^{\nu_2(k+2)+3}}$, we get

$$\nu_2\big((q-1)(q^{k+2}-1)\big) = \nu_2(k+2) + 3. \tag{27}$$

In total, this gives a rather detailed description of the zeta values $\zeta(-1-k)$: We now know that the (reduced) denominator of $\zeta(-1-k)$ is a product of prime numbers p with $p-1 \mid k+2$, and we have upper bounds for their exponents. But still, it remains to ask *how good* this description is. For instance, is it always the case that *all* primes p with $p-1 \mid k+2$ appear in the denominator of $\zeta(-1-k)$ (i.e., with positive exponents)?
To answer this question, we shall finally discuss another (actually by far more elementary) approach to studying the zeta values $\zeta(-1-k)$, namely by expressing them in terms of Bernoulli numbers. Since the latter are quite well-understood, this allows us to test the quality of our results.

9 Concluding remarks

As already indicated, it is possible to express the zeta values

$$\zeta(-1-k) = -\frac{B_{k+2}}{k+2}$$

in terms of Bernoulli numbers, see [7, Thm. 2, p. 231], which are defined by means of the Taylor series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

This allows us to translate properties of Bernoulli numbers into properties of the $\zeta(-1-k)$. For instance, for any $n \in \mathbb{N}$ and prime number p such that $p-1 \nmid n$, *Adam's theorem* (cf. [7, Prop. 15.2.4]) asserts that

$$\frac{B_n}{n} \in \mathbb{Z}_{(p)},$$

proving (25). Moreover, it is known that the (reduced) denominator of B_n for even $n \in \mathbb{N}$ is precisely $\prod_{p-1|n} p$, i.e., the product over all primes p such that $p-1 \mid n$; see [7, p. 233]. Combining this with Adam's theorem, we see that the (reduced) denominator of $\zeta(-1-k)$ is given by

$$\prod_{p-1|k+2} p^{\nu_p(k+2)+1}.$$

This shows that the upper bound (26) is sharp, while (27) is not optimal, but still quite close to being best possible.

Finally, we shall make clear that these comments also suggest that none of our results (25), (26), (27) is new. However, it is important to point out that the "polylogarithmic approach" presented here is applicable to a by far larger class of zeta and L-values; it was only for the sake of simplicity that we restricted ourselves to Riemann's ζ in the current article. The reader is particularly recommended to consult [1, §5] or [9] for instances of such applications.

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