Cohomological approaches to rationality of L-values

Julio de Mello Bezerra

University of Regensburg
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Riemann Zeta

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$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \Gamma(s) := \int_0^{\infty} e^{-u} u^{s-1} du,$$

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Note that, for $\lambda \in \mathbb{R}_+$,

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after using the change of variables $u = \lambda t$. Then one can set $\lambda = n$ and sum over all natural numbers:

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nt} t^{s-1} dt.$$

We may define $G(t) := \frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^{\infty} e^{-nt}$.

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Lemma

Under certain hypotheses on $f \in C^\infty(\mathbb{R}_{\geq 0})$, $M(f, s) := \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^{s-1} dt$ admits an analytic continuation to all of $\mathbb{C}$, and for $k \in \mathbb{N}$, $M(f, -k) = (-1)^k (\frac{d}{dt})^k f(t)\big|_0$.

We also have that $\zeta(s) = \frac{1}{s-1} M(tG(t), s-1)$.

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We also have that $\zeta(s) = \frac{1}{s-1} M(tG(t), s-1)$.

$$\text{Therefore} \qquad \zeta(-k) = \frac{(-1)^k B_{k+1}}{k+1},$$

where $B_k \in \mathbb{Q}$ is the k-th Bernoulli number, defined by

$$\sum_{n=0}^\infty B_n \frac{t^n}{n!} = \frac{te^{-t}}{1-e^{-t}} = tG(t).$$

Dirichlet L-function and Hurwitz Zeta function

Analogously, for $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$ and $x \in [0, 1)$, we define

$$L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad \text{and} \quad \zeta(s, x) := \sum_{n=1}^{\infty} (n + x)^{-s},$$

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and conclude that

$$\zeta(-k, x) = -\frac{B_{k+1}(x)}{k+1} \quad \text{and} \quad L(-k, \chi) = -\sum_{a=1}^{N} \chi(a) N^k \frac{B_{k+1}(a/N)}{k+1},$$

where $B_n(x) \in \mathbb{Q}[x]$ is the n-th Bernoulli polynomial, defined by

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = t \frac{e^{tx}}{e^t - 1}.$$

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- For an ideal $\mathfrak{f} \subset \mathcal{O}_F$, $\mathfrak{I}_\mathfrak{f}$ is the group of fractional ideals coprime to $\mathfrak{f}$;
- $R_\mathfrak{f}$ is the *ray class group modulo* $\mathfrak{f}$, which is a quotient of $\mathfrak{I}_\mathfrak{f}$;

The Hecke L-function

For $\mathfrak{f} \subset \mathcal{O}_F$ we have the finite Hecke characters with conductor $\mathfrak{f}$

$$\chi \colon R_{\mathfrak{f}} \to \mathbb{C}^{\times}$$

giving rise to the *Hecke L-function*

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s}.$$

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For $\mathfrak{b} \subset \mathcal{O}_F$ we define $\zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{[\mathfrak{a}] \sim [\mathfrak{b}]} N(\mathfrak{a})^{-s}$ so we have

$$L(\chi, s) = \sum_{[\mathfrak{b}] \in R_{\mathfrak{f}}} \chi(\mathfrak{b}) \zeta(\mathfrak{b}, \mathfrak{f}, s).$$

We want to show that the values $L(\chi, -k) \in \mathbb{Q}(\chi)$ are rational up to the image of the character χ for $k \geq 0$.

The Siegel-Klingen Theorem

Theorem (Siegel-Klingen)

With the definitions as above,

$$\zeta(\mathfrak{b}, \mathfrak{f}, 1-k) \in \mathbb{Q} \quad \forall k \geq 1 \in \mathbb{Z}.$$

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Sketch of proof: Let $f(z)$ be an $SL_2(\mathbb{Z})-$Eisenstein series, $z \in \mathbb{H}$, of weight k with q-expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}$$

and show that there are integers $c_{k,0}, \ldots, c_{k,r}$ such that

$$c_{k,0}a_0 + c_{k,1}a_1 + \cdots + c_{k,r}a_r = 0, \quad c_{k,0} \neq 0$$

where $r := r(k) := \dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z}))$.

The Siegel-Klingen Theorem

Now show that for each $k \geq 1$ there is an Eisenstein series $f(z)$ such that $a_0 = C\zeta(\mathfrak{b}, \mathfrak{f}, 1-k)$ with $C, a_1, \ldots, a_r \in \mathbb{Q}$. The previous result implies that $a_0 \in \mathbb{Q}$ as desired.

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Example: The classical Eisenstein series

$$\mathbb{E}_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} (mz+n)^{-k} =$$

$$= \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and even $k \geq 4$ gives

$$\zeta(1-k) = -\frac{2}{c_{k,0}} \sum_{j=1}^{r} \sigma_{k-1}(j) c_{k,j} \in \mathbb{Q}.$$

The Shintani Method

Let $\Delta := \mathcal{O}_{F+}^{\times}$ be the set of totally positive units of $\mathcal{O}_F$.

For $\mathfrak{f} \subset \mathcal{O}_F$ we have the additive characters

$$\xi \in U[\mathfrak{f}] := \mathrm{Hom}(\mathcal{O}_F/\mathfrak{f}, \mathbb{C}^{\times}) \setminus \{1\},$$

defining the *Lerch zeta function*

$$\mathcal{L}(\xi\Delta, s) := \sum_{\alpha \in \Delta_\xi \backslash \mathcal{O}_{F+}} \xi(\alpha) N(\alpha)^{-s}.$$

where $\Delta_\xi := \{\varepsilon \in \Delta \mid \xi(\varepsilon\alpha) = \xi(\alpha), \quad \forall \alpha \in \mathcal{O}_F\}$.

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Proposition

Assume that $|R_1| = 1$, then for $\chi \neq 1$ we have

$$L(\chi, s) = \sum_{\xi \in \Delta \backslash U[\mathfrak{f}]} c_\chi(\xi) \mathcal{L}(\xi\Delta, s).$$

The Shintani Decomposition

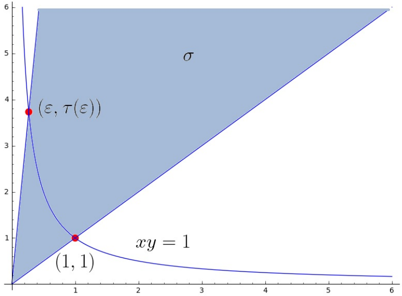
A cone generated by $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathcal{O}_{F+}^m$ is any set of the form

$$\sigma_{\boldsymbol{\alpha}} := \{x_1\alpha_1 + \ldots + x_m\alpha_m \mid x_1, \ldots, x_m \in \mathbb{R}_{\geq 0}\}.$$

Shintani's Unit Theorem

There is a set of cones Φ such that $\mathbb{R}_+^g = \bigcup_{\sigma \in \Phi} \sigma$, Φ is closed under the Δ action and the quotient $\Delta \backslash \Phi$ is a finite set.

When $g = 2$:



The Shintani Zeta Function

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and σ a cone, we define the *Shintani zeta function*:

$$\zeta_\sigma(\xi, s) = \sum_{\alpha \in \breve{\sigma} \cap \mathcal{O}_F} \xi(\alpha) N(\alpha)^{-s}.$$

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Theorem (Shintani, Cassou-Noguès)

$\zeta_\sigma(\xi, s)$ admits a meromorphic continuation to $\mathbb{C}$ and there exists a meromorphic function $\mathcal{G}_\sigma(t)$ on $F \otimes \mathbb{C}$ such that for each $k \in \mathbb{N}$,

$$\partial^k \mathcal{G}_\sigma(t)\Big|_{t_\xi} = \zeta_\sigma(\xi, -k).$$

Furthermore, if Φ is a Shintani decomposition, then

$$\mathcal{L}(\xi\Delta, s) = \sum_{\sigma \in \Delta_\xi \backslash \Phi} \zeta_\sigma(\xi, s).$$

Equivariant Sheaf Cohomology

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- A Δ-*equivariant sheaf* is an $\mathscr{O}_U$-module $\mathscr{F}$ together with a family of isomorphisms

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- The *equivariant cohomology* $H^m(U/\Delta, -)$ is defined to be the m-th right derived functor of $\Gamma(U/\Delta, -)$.

Specialization Map

For each finite character $\xi \in U$, let $\mathbb{Q}(\xi)$ be the number field obtained by adjoining the image of ξ to $\mathbb{Q}$. Then each torsion point $\xi \in U$ can be identified with $\operatorname{Spec} \mathbb{Q}(\xi)$.

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$\xi = \operatorname{Spec} \mathbb{Q}(\xi)$ is a Δ_ξ-scheme, and the inclusion $\xi \to U$ is compatible with the inclusion $\Delta_\xi \subset \Delta$, so we have the specialization map

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Furthermore, there exists a canonical isomorphism

$$H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \xrightarrow{\cong} \mathbb{Q}(\xi)$$

which allows us to identify them as $\mathbb{Q}$ vector spaces.

Canonical Decomposition

Theorem (Bannai, Hagihara, Yamada, Yamamoto)

There exist a canonical class $\mathcal{G} \in H^{g-1}(U/\Delta, \mathscr{O}_U)$ and a homomorphism $\partial\colon H^m(U/\Delta, \mathscr{O}_U) \to H^m(U/\Delta, \mathscr{O}_U)$ induced by a differential operator on $\mathscr{O}_U$, such that

$$\xi^*(\partial^k \mathcal{G}) = \mathcal{L}(\xi\Delta, -k)$$

through the identification $H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \xrightarrow{\cong} \mathbb{Q}(\xi)$.

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Assume that $|R_1| = 1$, then for $\chi \neq 1$ we have

$$L(\chi, -k) = \sum_{\xi \in U[\mathfrak{f}]/\Delta} c_\chi(\xi)\xi^*(\partial^k \mathcal{G}) \in \mathbb{Q}(\chi).$$

Remarks

- If the number field K is not totally real, then $L(\chi, -k) = 0$ for all $k \in \mathbb{Z}_+$ and χ a finite Hecke character.

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- If $|R_1| > 1$ we may instead use the $F_+^\times$-scheme $\widetilde{\mathbb{T}} := \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$, where $\mathbb{T}^{\mathfrak{a}} := \mathrm{Hom}(\mathfrak{a}, \mathbb{C}^\times)$, to get a canonical representation of the Hecke L-function in terms of the cohomology class. To prove the rationality of the special L-values, one may simply sum over the narrow classes.

A Topological Torus

Let $\mathfrak{b}$ and $\mathfrak{f}$ be integral ideals as before and consider the following zeta function

$$\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, s) := \sum_{\alpha \in (1+\mathfrak{f}\mathfrak{b}^{-1})^{+}/\Gamma} N(\alpha)^{-s}$$

where $\Gamma := \{\varepsilon \in \Delta \mid \varepsilon \in 1 + \mathfrak{f}\mathfrak{b}^{-1}\}$.
Then

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$$\zeta(\mathfrak{b}, \mathfrak{f}, s) = N(\mathfrak{b})^{-s} \zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, s).$$

Note that $\mathfrak{f}\mathfrak{b}^{-1}$ is a $\mathbb{Z}$-module of rank g, so it defines a full lattice inside $F \otimes \mathbb{R}$. Therefore we may define the g-dimensional topological torus

$$T := (F \otimes \mathbb{R})/\mathfrak{f}\mathfrak{b}^{-1} \cong (\mathbb{S}^1)^g,$$

$$0 \to \mathfrak{f}\mathfrak{b}^{-1} \to (F \otimes \mathbb{R}) \xrightarrow{\pi} T \to 0.$$

The Logarithm Sheaf

Let A be a $\mathbb{Q}$-algebra and let $R_A := A[[Y-1]] \cong A[[t]]$ be the power series ring on $Y = Y_1, \ldots, Y_g$ variables with isomorphism given by $Y_i \mapsto e^{t_i}$.

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We define the *Logarithm sheaf* on T as the sheaf assigning to each open $U \subset T$

$$\mathcal{L}\text{og}(U) = \{f : \pi^{-1}(U) \to R_A \mid f(x+l) = Y^{-l} f(x) \;\; \forall l \in \mathfrak{f}\mathfrak{b}^{-1}, x \in \pi^{-1}(U)\}.$$

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Note that for the Hurwitz Zeta function

$$\zeta(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(t,x) t^{s-1} dt,$$

the generating function is a section of this sheaf

$$G(t,x) := \frac{e^{-xt}}{1-e^{-t}} = \sum_{n=1}^{\infty} e^{-(n+x)t}.$$

Polylogarithm Class

The logarithm sheaf $\mathcal{L}\mathrm{og}$ is a Γ-equivariant sheaf and we can define a unique cohomology class

$$pol \in H^{g-1}((T \setminus \{0\})/\Gamma; \mathcal{L}\mathrm{og})$$

called the *topological polylogarithm*.

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When specialized to a torsion point $h \in T$ stabilized by Γ, this class yields

$$Eis(h) := h^* pol \in H^{g-1}(\Gamma, \mathcal{L}\text{og}_h) = H^{g-1}(\Gamma, R_A)$$

the *Eisenstein class* associated to h. There exists further an evaluation map

$$ev : H^{g-1}(\Gamma, R_A) \longrightarrow (R_A)_\Gamma.$$

Rationality of Coefficients

For any subring $A \subset \mathbb{C}$, we have a commutative diagram

$$\begin{array}{ccc} H^{g-1}(\Gamma, R_A) & \xrightarrow{ev} & (R_A)_\Gamma \\ \downarrow & & \downarrow \\ H^{g-1}(\Gamma, R_{\mathbb{C}}) & \xrightarrow[ev]{} & (R_{\mathbb{C}})_\Gamma. \end{array}$$

Rationality of Coefficients

For any subring $A \subset \mathbb{C}$, we have a commutative diagram

$$\begin{array}{ccc} H^{g-1}(\Gamma, R_A) & \xrightarrow{ev} & (R_A)_\Gamma \\ \downarrow & & \downarrow \\ H^{g-1}(\Gamma, R_\mathbb{C}) & \xrightarrow[ev]{} & (R_\mathbb{C})_\Gamma. \end{array}$$

For $A = \mathbb{C}$, the polylogarithm class $pol_\mathbb{C}$ can be given explicitly using certain generalized Eisenstein series.

Furthermore, $(R_\mathbb{C})_\Gamma = \mathbb{C}[[w]] \subset \mathbb{C}[[t]]$ for $w := t_1 \cdots t_g$.

Rationality of L-values

Theorem (Beilinson-Kings-Levin)

Using our previous definitions, we have

$$ev(1^* pol_{\mathbb{C}}) = (-1)^{g-1} \sum_{k \geq 0} \zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, -k) \frac{w^k}{(k!)^g} \in \mathbb{C}[[w]].$$

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From this theorem and the previous commutative diagram, it follows from setting $A = \mathbb{Q}$ that $\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, -k) \in \mathbb{Q}$ for $k \geq 0$.

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Remark: This approach should give a framework that generalizes the previous approaches.

Integrality

Let $\mathfrak{c}$ be an integral ideal coprime to $\mathfrak{f}\mathfrak{b}^{-1}$. By setting $A := \mathbb{Z}[\frac{1}{N(\mathfrak{c})}]$, a careful modification of the previous theorem yields:

Theorem (Deligne-Ribet, Cassou-Noguès)

For $\mathfrak{f} \neq \mathcal{O}_F$ and $k \geq 0$ one has

$$(N\mathfrak{c})^{1+k}\zeta(\mathfrak{b}, \mathfrak{f}, -k) - \zeta(\mathfrak{b}\mathfrak{c}, \mathfrak{f}, -k) \in N\mathfrak{b}^k\mathbb{Z}[N(\mathfrak{c})^{-1}].$$

If $\mathfrak{f} = \mathcal{O}_F$ the result holds for $k \geq 1$.

This type of theorem is what is known as an "integrality result". It allows us to define p-adic L-functions!

p-adic L-functions

Suppose now that p is an odd prime and χ has values in $\overline{\mathbb{Q}_p}^{\times}$.

Theorem (Deligne-Ribet, Cassou-Noguès, Barsky)

There exists an unique continuous function $L_p(\chi, s) : \mathbb{Z}_p \to \overline{\mathbb{Q}_p}$ such that

$$L_p(\chi, 1-k) = L(\chi\omega^{-k}, 1-k) \prod_{\mathfrak{p}|p} (1 - \chi\omega^{-k}(\mathfrak{p}) N\mathfrak{p}^{k-1}) \quad \forall k \geq 0,$$

where ω is the Teichmüller character and the product is over the primes of F above p.

This theorem can be deduced from the previous integrality results using the technique of p-adic interpolation developed by Kubota-Leopoldt.