Introduction to STACK

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Warwick Maths Teaching and Learning Seminar
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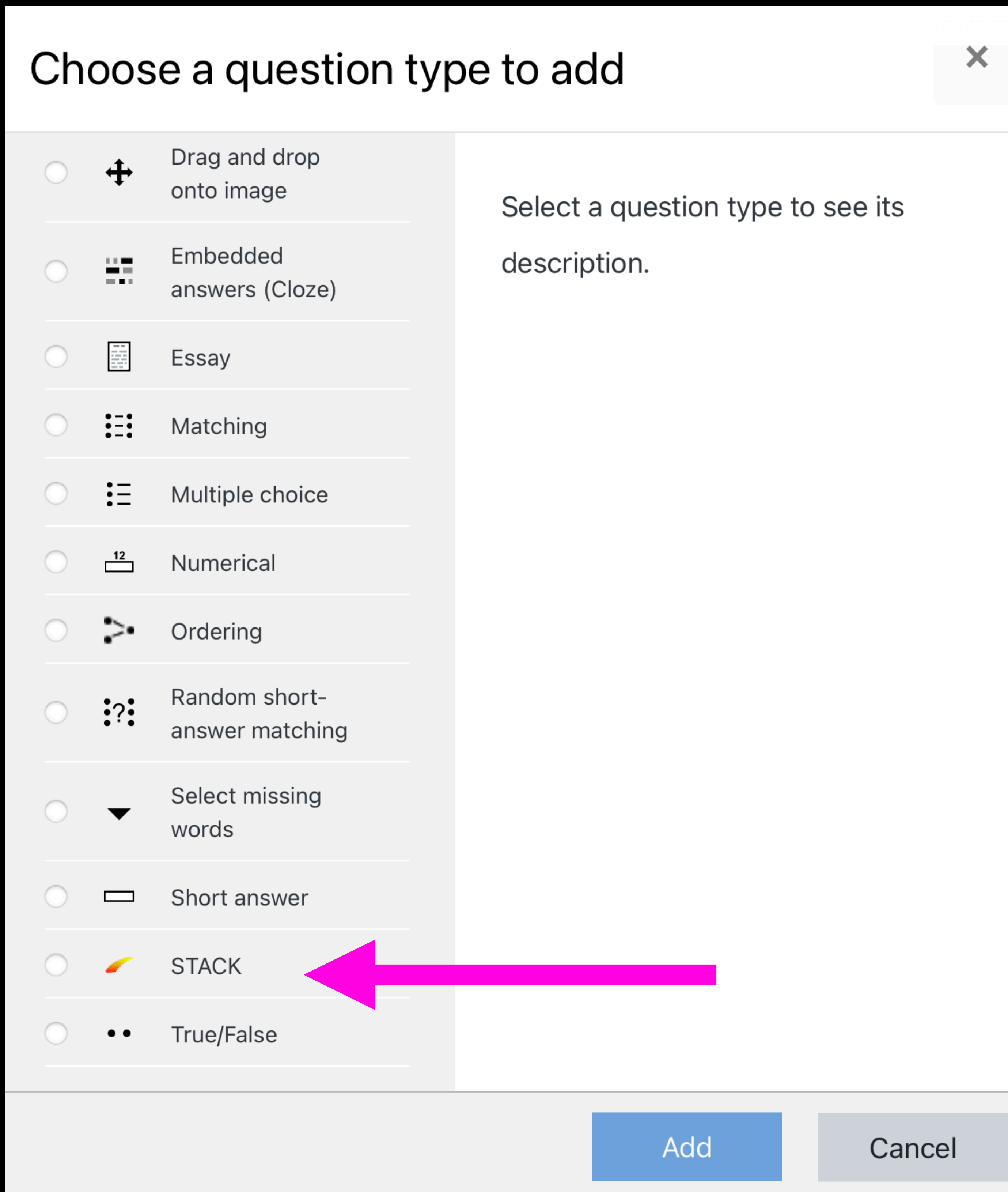
What is STACK?



- A Moodle quiz question type tailored for maths quizzes
- Created by Chris Sangwin at Edinburgh https://stack-assessment.org
- Allows variable randomisation, LaTeX-style questions, marking with Maxima CAS.
- Open source!
- At Warwick, lives on MoodleX https://moodlex.warwick.ac.uk

Why use STACK?

Boring 😩	With STACK 😮
Multiple-choice questions	Write down equations, expressions…
Everyone gets the same questions	Variables/figures can be randomised with every attempt
Figures are static	Embed interactive figures into questions
Explain proofs in lecture	Make students engage with proofs

Case study: Analysis II (20/21)

• Planning	• June 2020
• Coding+Testing	
• Launch	• Jan 2021
• Results	
• Feedback	• April 2021

MA131 Analysis II Resources

Lecture Notes

Lecture notes will be distributed before each lecture and will be available to download here.
My notes are "fill-in-the-blanks" type. We fill in the blanks together in class.
Printed notes are given out at the beginning of lectures, after which they will be available from the filing cabinet outside the UG office.

- Week 1: Continuity I (blank/annotated)
- Week 2: Continuity II (blank/annotated)
- Week 3: Continuity on an interval (blank/annotated)
- Week 4: Limits (blank/annotated)
- Week 5: Inverse functions + Differentiation I (blank/annotated)
- Week 6: Differentiation II (blank/annotated)
- Week 7: MVT + L'Hôpital's rules I (blank/annotated)
- Week 8: Taylor's Theorem (blank/annotated)
- Week 9: Radius of convergence (blank/annotated)
- Week 10: Series differentiation (blank/annotated)
- Exam revision checklist

Year 1 Modules

Year 1 regs and modules
G100 G103 GL11 G1NC

Year 2 Modules

Year 2 regs and modules
G100 G103 GL11 G1NC

Year 3 Modules

Year 3 regs and modules
G100 G103

Year 4

19/20

20/21

Assignments

There will be 8 assessed assignment sheets. The assessed problems
will be available to download below. Please submit your work to the
opposite the undergraduate office.

The best 7 of the 8 marks will count for 7.5% of the 24CAT Analysis (

Each assignment sheet will have three sections. The first (A) section
to familiarise you with the material covered in lectures. The second (
hand in. The final (C) section consists of additional exercises, which m
interesting.

- Sheet 0 - no need to hand in, but forms the basis of discussion in
- Sheet 1 - to be handed in by 12 noon, Thursday 16th January.
- Sheet 2 - to be handed in by 12 noon, Thursday 23rd January.
- Sheet 3 - to be handed in by 12 noon, Thursday 30th January.
- Sheet 4 - to be handed in by 12 noon, Thursday 6th February.
- Sheet 5 - to be handed in by 12 noon, Thursday 13th February.
- Sheet 6 - to be handed in by 12 noon, Thursday 20th February.
- Sheet 7 - to be handed in by 12 noon, Thursday 27th February.
- Sheet 8 - to be handed in by 12 noon, Thursday 5th March.
- Sheet 9 - not assessed, but contains examinable material.

Assignments:

Quiz 0 (syntax training)
Quiz 0.5 (revision of Analysis I)
Quiz 1
Quiz 2 + Sheet I
Quiz 3
Quiz 4+ Sheet II
Quiz 5
Quiz 6
Quiz 7 + Sheet III
Quiz 8
Quiz 9 + Sheet IV

Flat 5% for passing 6/9 Quizzes
10% for Sheets.

What I did

- My background pre-Covid: MapleTA/Möbius (expensive)
- Attended STACK workshop in June 2020
- Recruited a summer intern (Kate Harrison) to work with me over 4 weeks, coding 40+ multi-part questions
- Infinite attempts, each building on the last, pass mark ~80%
- We asked George Kinnear and Chris Sangwin (Edinburgh) a lot of questions
- 2 conference talks and a conference proceeding published with Kate

Highlights from Analysis II quizzes (20/21)

This result is essential for Analysis II. Please study it carefully.
Complete the proof by filling in the blanks.

Scaffolded proof + drop downs

Archimedean Property: For all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

Proof:

1) We will prove by contradiction.
Suppose, on the contrary, that [∃] [x ∈] $\mathbb{R}$, [∀] $n \in \mathbb{N}, n \leq x$.

In other words, x is an upper bound of $\mathbb{N}$.

2) The non-empty set $\mathbb{N}$ therefore has a [(No answer given) / ✓ supremum / infimum] u.

3) Since $u - 1 < u$, we know $u - 1$ is not [an upper bound] of $\mathbb{N}$ so there exists $m \in \mathbb{N}$ with $u - 1$ [<] m.

4) Therefore, u [<] $m + 1$ and since $m + 1 \in \mathbb{N}$, this contradicts the fact that u is [an upper bound].

Thus, the proof is complete.

Further question

Which step of the proof uses the Completeness property? [2]

Proof comprehension

Randomised mystery words

Books are helpful!

Books can often give you a different perspective on a topic from what your lecturer presents in class.
This question is my attempt to make you at least take a quick peek at the books I recommended for this module. My hope is that you will keep browsing, and come back to them again in the coming weeks.

a) In the main textbook for this course, written by Bartle and Sherbert (4th edition), the work of which famous mathematician is discussed in detail on page 103 ?

Gottfried Leibniz

b) Which theorem is discussed on page 116 of '*How to think about Analysis*' by Lara Alcock?
[*Hint: Look for "***Theorem (...)***" on the page and type in* _all_ *the words in the parentheses.*]

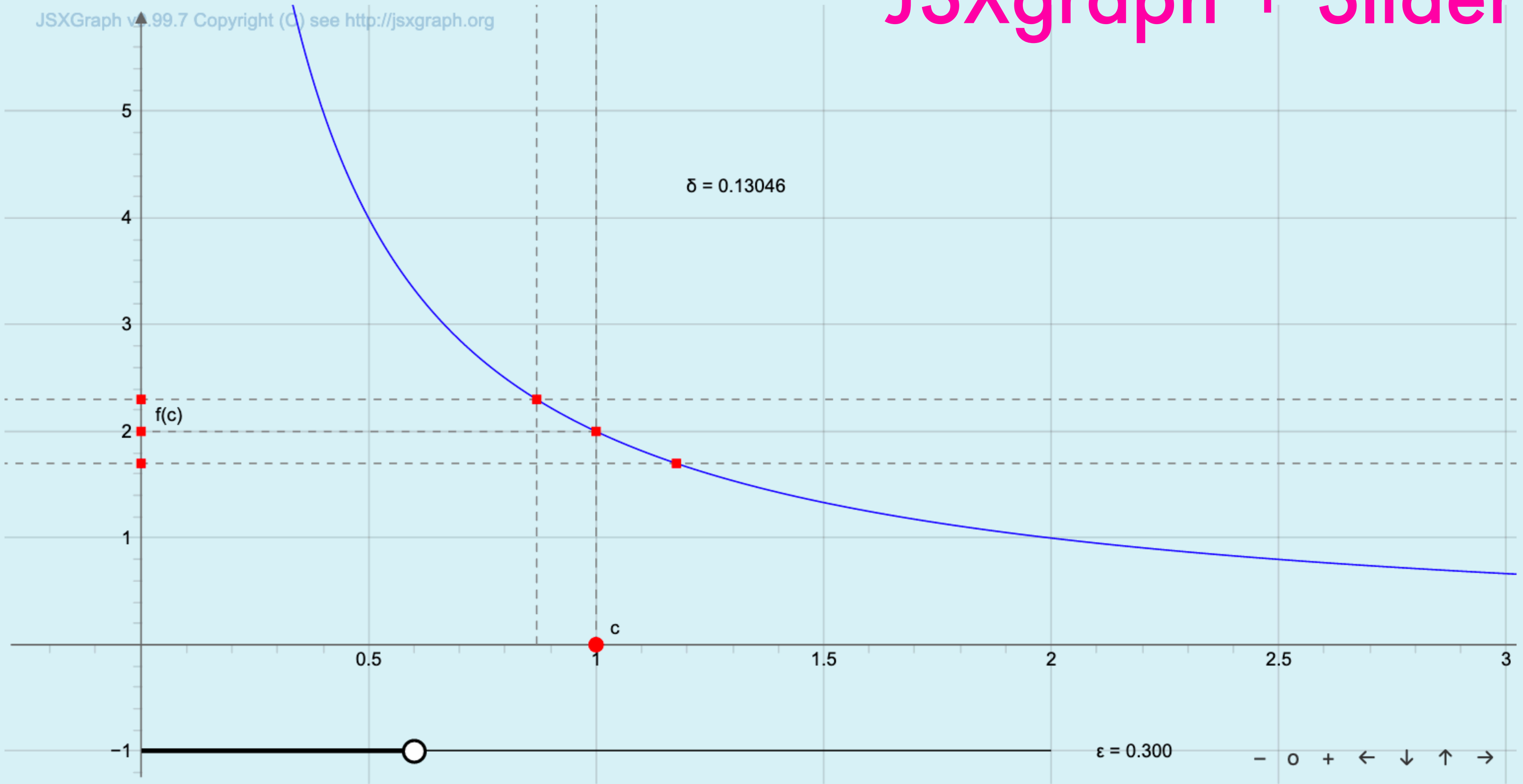
integral test

You may or may not get on with these books (which is ok - everyone likes different styles of writing and explanation). However, I really hope that you *do* find a book that you like - not just for Analysis, but for every module.

Here is a visualisation of the ϵ-δ definition for the continuity of a function f at the point $x = c$.

The curve $y = f(x)$ is shown in blue.

JSXgraph + Slider



You can use the slider to adjust these parameters:
- The value of ϵ
- The position of the point c.

Note that the slider has a tolerance of around 5×10^{-4}, and the numbers displayed are dependent on browsers.

a) The function is continuous at $x = 1$.
For each of the following values of ϵ, write down the largest value of δ such that:

$$|x - 1| < \delta \implies |f(x) - f(1)| < \epsilon.$$

(If you can't adjust the parameters to exactly these values, just use the nearest possible values.)

Give your answers to **2 decimal places**.

(i) $\epsilon = 0.3$, $\delta =$ 0.13

(ii) $\epsilon = 0.6$, $\delta =$ 0.23

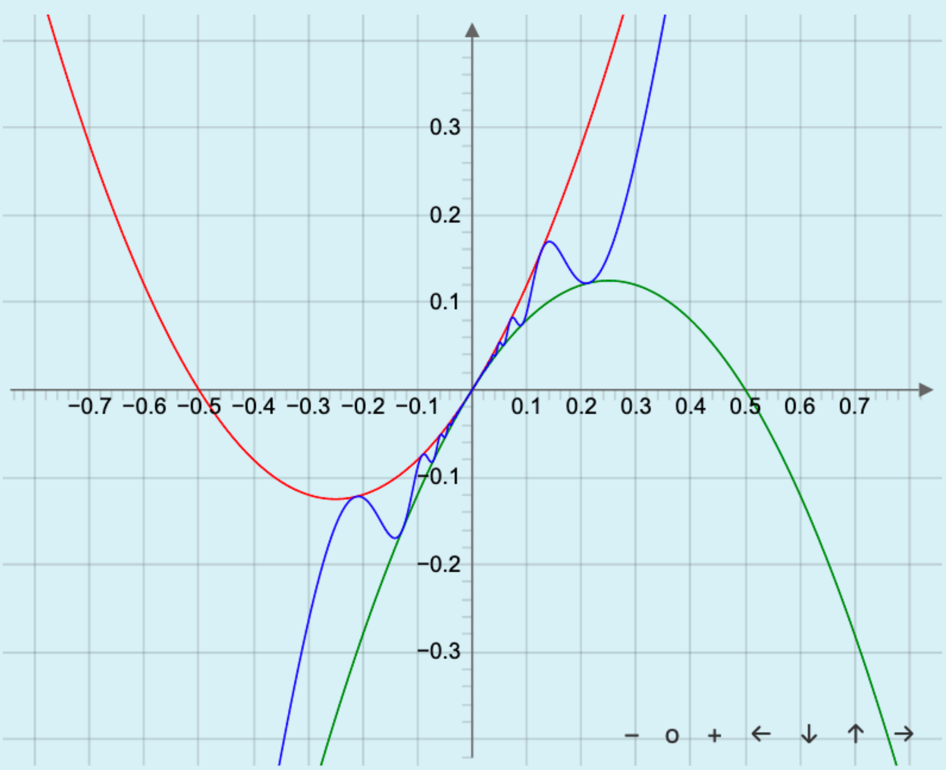
We are going to study a function, $f\colon \mathbb{R} \to \mathbb{R}$, with the following strange property:

> f is differentiable at $c \in \mathbb{R}$ and $f'(c) > 0$, yet there exists no neighbourhood of c in which f is strictly increasing.

(You should find this somewhat disturbing.)

One function with this property is:

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



The above graph shows the function $y = f(x)$ (the blue curve), which is bounded between the upper (red) curve and the lower (green) curve. (Try the zoom in/out and shift buttons).

JSXgraph zoom

Give the equations of the red and green curves.

Red: $y =$ ______

Green: $y =$ ______

Now we will show that the function f satisfies the strange property.

We start by proving that $f'(0) > 0$.

From the definition of a derivative,

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0}(\text{?+?*?*sin(1/x)}) \\ &= \text{a number}. \end{aligned}$$

Syntax hints

In the last step, we use the fact that $-2x \leq 2x\sin\left(\frac{1}{x}\right) \leq 2x$, and let $x \to 0$ and apply the ______ theorem.

2) We now prove that f is not locally increasing at $x = 0$.

2i) f is locally increasing at 0 iff $\forall \delta > 0$, (No answer given) ≥ 0 in $(-\delta, \delta)$.

2ii) Hence, it suffices to show that $\exists c \in (-\delta, \delta)$ such that (No answer given).

2iii) Now,

$$f'(x) = \begin{cases} ? & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Fill in the missing derivative. ______

2iv) By the ______ property of real numbers, $\forall \delta > 0, \exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < 2\pi\delta$.

2v) Let $c = \frac{1}{2\pi n} \in (0, \delta)$. Then $f'(c) =$ ______.

Hence, the function f indeed satisfies our counterintuitive property.

This is one reason why analysis is so important: our intuition can be misleading!

Recall the following from school mathematics:

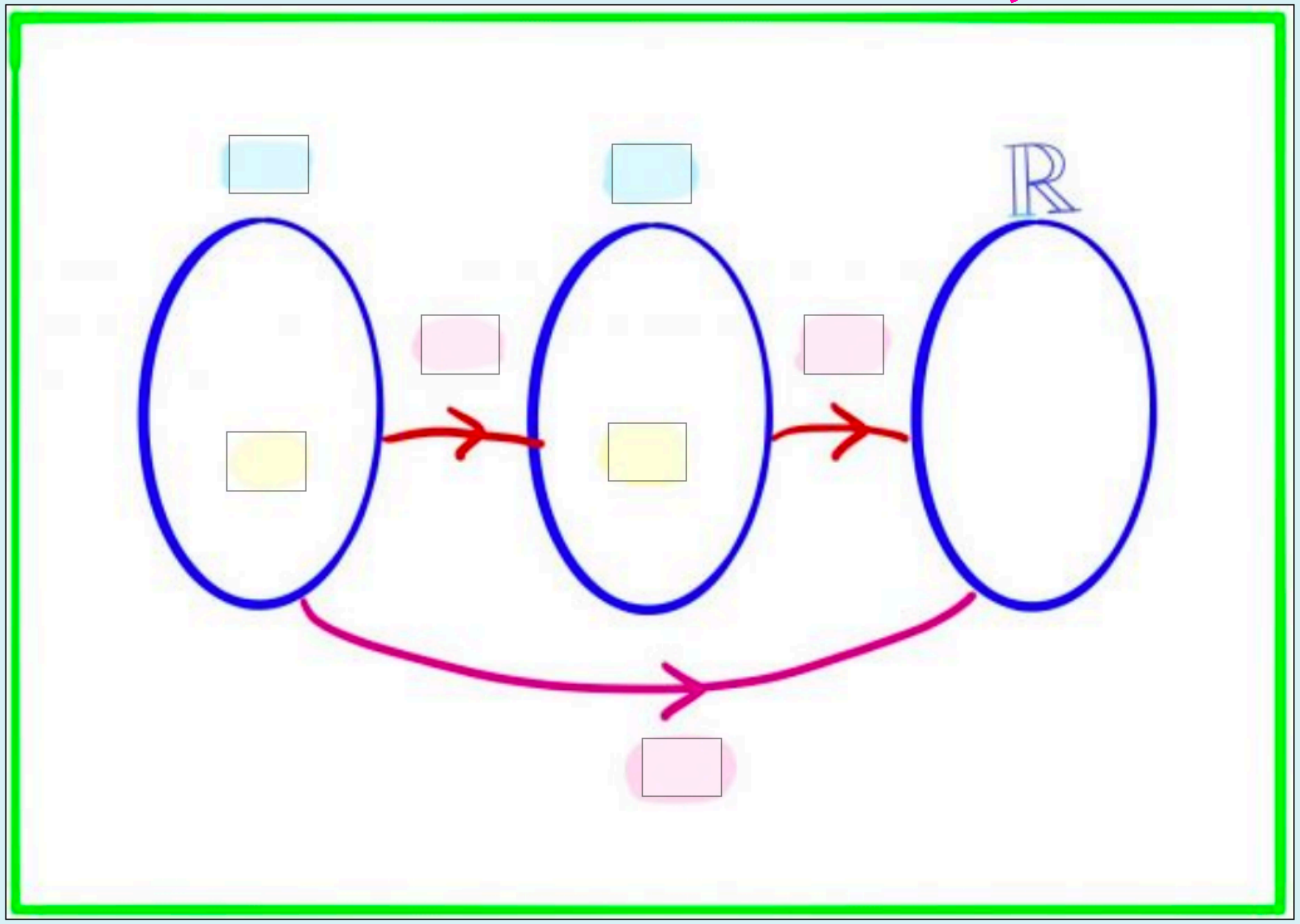
The Chain Rule: Let $g\colon I \to \mathbb{R}$ and $f\colon J \to \mathbb{R}$, where I and J are intervals, and $f(J) \subseteq I$. Let $c \in J$. Suppose f is differentiable at c and g is differentiable at $d = f(c)$, then the composition $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(d) \cdot f'(c).$$

In the picture below, drag the following objects into the correct positions to give a visualisation of the function composition in the Chain Rule. (This will help us with the proof below.)

Drag and drop onto jpg (non STACK)

[Hint: Blue= Set, Pink= Function, Orange= Number]



g | d | J | f ∘ g | g ∘ f | I | f | c

Tidy STACK question tool | Question tests & deployed variants

We will now prove the Chain Rule using Carathéodory's Theorem.

Fill in the blanks (STACK)

a) Fill in the blanks in the proof below.

Proof:

1) Since f is differentiable at c, $\exists\ \varphi_1 \colon J \to \mathbb{R}$, continuous at c, such that $\forall x \in J,\ f(x) - f(c) =$ phi_1(x)*(x-c)

and $f'(c)=$ phi_1(c) .

2) Since g is differentiable at d, $\exists\ \varphi_2 \colon I \to \mathbb{R}$, continuous at d, such that $\forall y \in I,\ g(y) - g(d) =$ phi_2(y)*(y-d)

and $g'(d) =$ phi_2(d) .

3) Let $y = f(x)$. Then

$$g(f(x)) - g(f(c)) = \varphi_2(f(x)) \cdot [f(x) - f(c)]$$

and so $g \circ f(x) - g \circ f(c) = \varphi_2 \circ f(x) \cdot$ phi_1(x)*(x-c) .

4) Hence, where $\varphi(x) = \varphi_2 \circ f(x) \cdot \varphi_1(x)$, we know that φ is continuous at c and so $g \circ f$ is differentiable at c.

5) We therefore have $(g \circ f)'(c) = \varphi(c) = \varphi_2 \circ f(c) \cdot$ phi_1(c) $= g'(d) \cdot f(c)$.

Further questions:

Proof comprehension

b) Which of the above steps is a consequence of Carathéodory's Theorem? 1 and 2

c) Which step of the proof is a consequence of the Algebra of Continuous Functions? 4

Suppose that there exists a function $E\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} E'(x) = E(x) & (A) \\ E(0) = 1 & (B) \end{cases}$$

Prove that $E(x)$ is unique.

Proof comprehension (STACK) followed by...

Proof:

1) Firstly, suppose $\exists$ 2 differentiable functions, $E_1, E_2\colon \mathbb{R} \to \mathbb{R}$ which satisfy conditions A and B.

2) If we consider the derivative $\left(\frac{E_1(x)}{E_2(x)}\right)'$ then, by the [ ] rule for differentiation, we have

$$\left(\frac{E_1(x)}{E_2(x)}\right)' = \frac{E_2(x)E_1'(x) - E_1(x)E_2'(x)}{E_2^2(x)}.$$

3) Using (No answer given) ⬍, we therefore have

$$\left(\frac{E_1(x)}{E_2(x)}\right)' = \frac{E_2(x)E_1(x) - E_1(x)E_2(x)}{E_2^2(x)} = 0.$$

4) Hence, step 3 $\implies$ $\frac{E_1(x)}{E_2(x)}$ is (No answer given) ⬍ on $\mathbb{R}$.

5) Consequently,

$$\frac{E_1(x)}{E_2(x)} = \frac{E_1(0)}{E_2(0)} = C$$

where $C =$ [ ] using (No answer given) ⬍.

6) We therefore have $E_1(x) = E_2(x)$ and so the function $E\colon \mathbb{R} \to \mathbb{R}$ is unique.

Further questions:

Place the following statements in the correct order to prove that $\forall r \in \mathbb{Q}$

$$E(r) = e^r,$$

where the constant $e \equiv E(1)$.

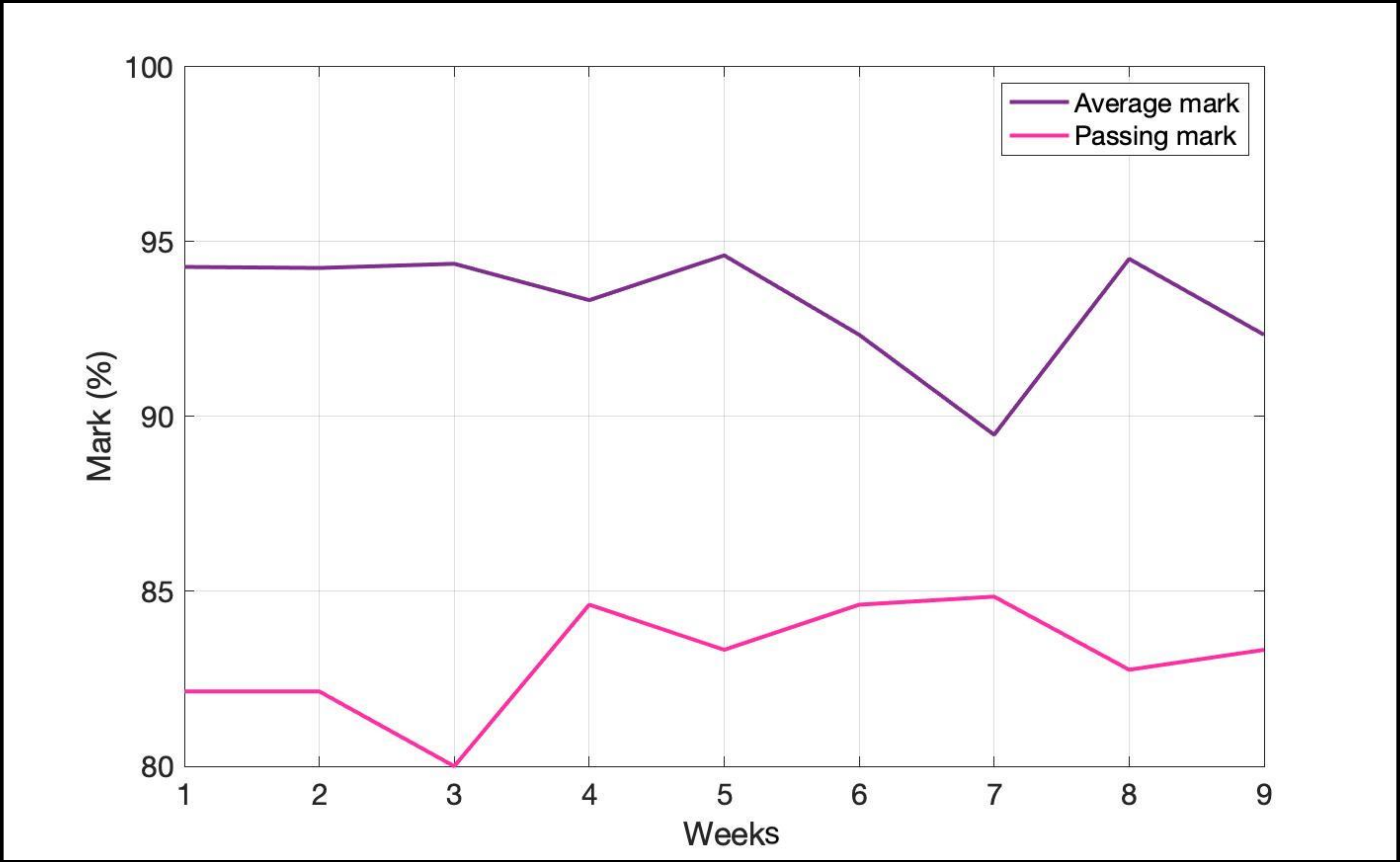
Ordering (non STACK)

- Thus $E(r) = e^r$ holds for all $r = \pm\frac{m}{n}$ where $m, n \in \mathbb{N}$.
- $E\left(-\frac{m}{n}\right) = \frac{1}{E(m/n)} = \frac{1}{e^{\frac{m}{n}}} = e^{-\frac{m}{n}}$.
- using the property $E(x+y) = E(x) \cdot E(y)$ inductively.
- Consequently, $E\left(\frac{1}{n}\right) = [E(1)]^{\frac{1}{n}} \implies E(r) = e^r$. Once again, the relation holds.
- If $r \in \mathbb{Z}$, $E(r) = E(1+1+1+\ldots+1) = E(1) \cdot E(1) \cdot E(1) \cdots E(1)$,
- Hence, the relation $E(r) = (E(1))^r = e^r$ for all $r \in \mathbb{Z}$.
- Hence, $E(1) = \left[E\left(\frac{1}{n}\right)\right]^n$.
- Hence, $E(r) = \left[E\left(\frac{1}{n}\right)\right]^m = \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}} = e^r$, and the relation also holds in this case.
- If $r = \frac{m}{n}$ where $m, n \in \mathbb{N}$, then $E(r) = E\left(\frac{m}{n}\right) = E\left(\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right)$.
- If $r = -\frac{m}{n}$ where $m, n \in \mathbb{N}$, we use the property $E(-x) = 1/E(x)$ to deduce that
- If $r = \frac{1}{n}$ where $n \in \mathbb{N}$, we have $E(1) = E\left(n \cdot \frac{1}{n}\right) = E\left(\frac{1}{n} + \frac{1}{n} + \frac{1}{n} \cdots + \frac{1}{n}\right)$.
- Combining these results with the fact that $E(0) = 1 = e^0$, the property $E(r) = e^r$ holds for all $r \in \mathbb{Q}$.

Results*

*no mitigation/discount taken into account.

Average marks VS pass marks



How many attempts?

Attempts	**1**, 2, 3, 4, 5, 6
Started on	Monday, 25 January 2021, 12:22 PM
State	Finished
Completed on	Monday, 25 January 2021, 12:40 PM
Time taken	17 mins 7 secs
Grade	**38.00** out of 40.00 (**95**%)
Feedback	Congratulations! You have passed.

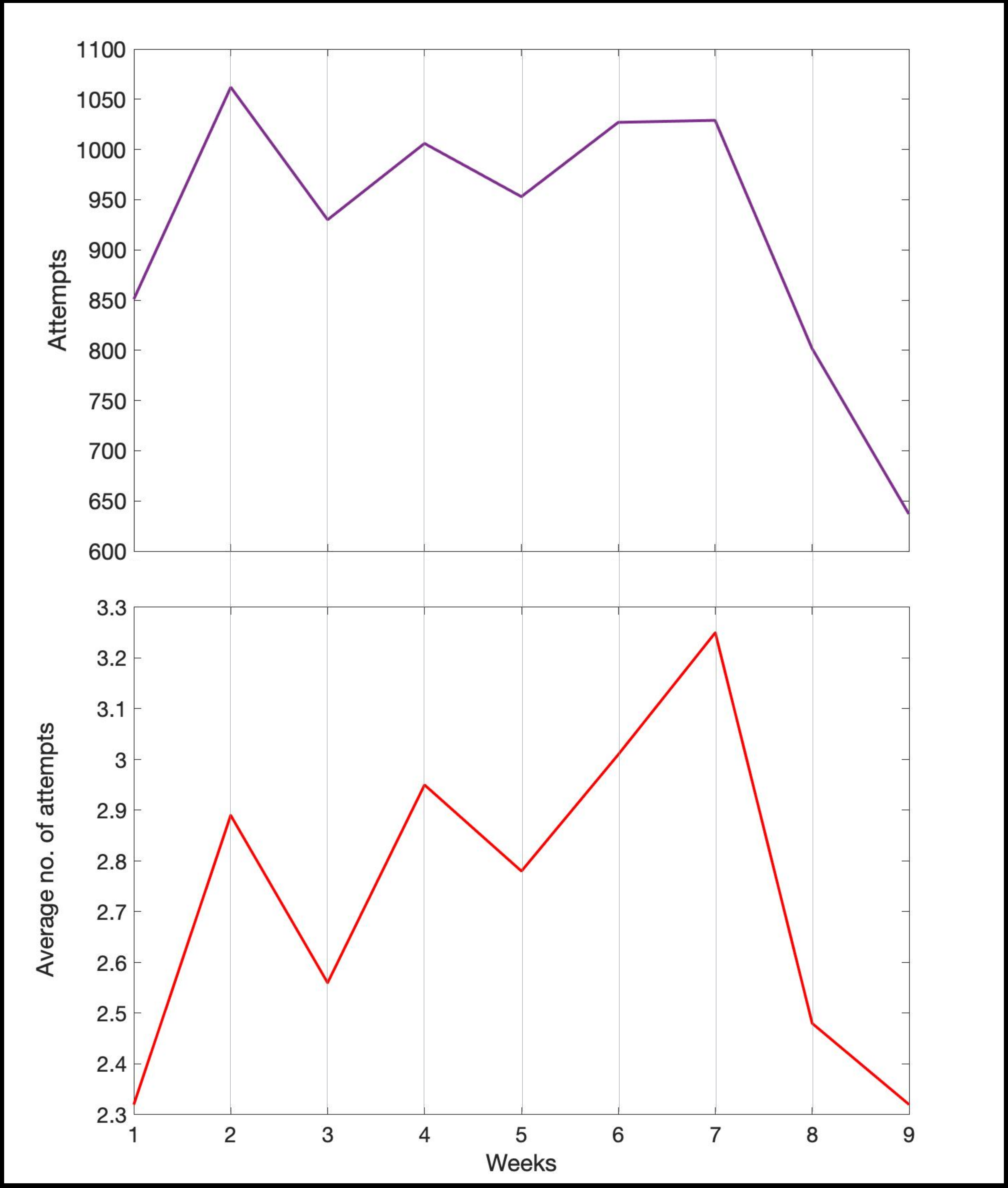
Passed first go but did 5 more times.

Attempts	1, 2, 3, 4, 5, 6, 7, **8**, 9
Started on	Friday, 26 February 2021, 2:57 PM
State	Finished
Completed on	Friday, 26 February 2021, 2:58 PM
Time taken	24 secs
Grade	**26.00** out of 33.00 (**79**%)
Feedback	You have not passed. Please try again!

Took 9 attempts to pass.

Extreme cases of multiple attempts.

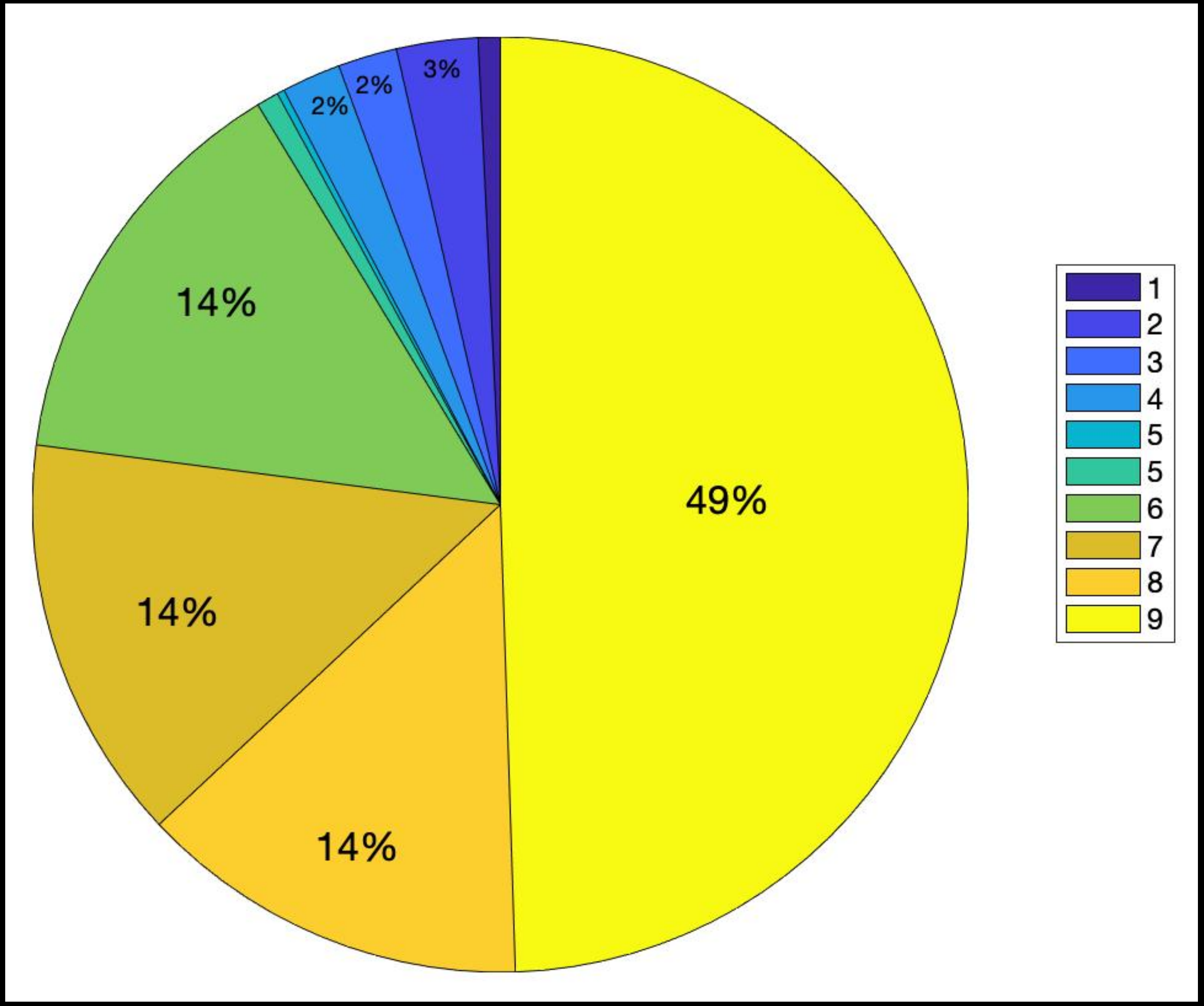
Total number of attempts



Students attempted 2-3 times on average

Are students relying on safety net? No

After all, they only need to pass 6 out of 9 quizzes.

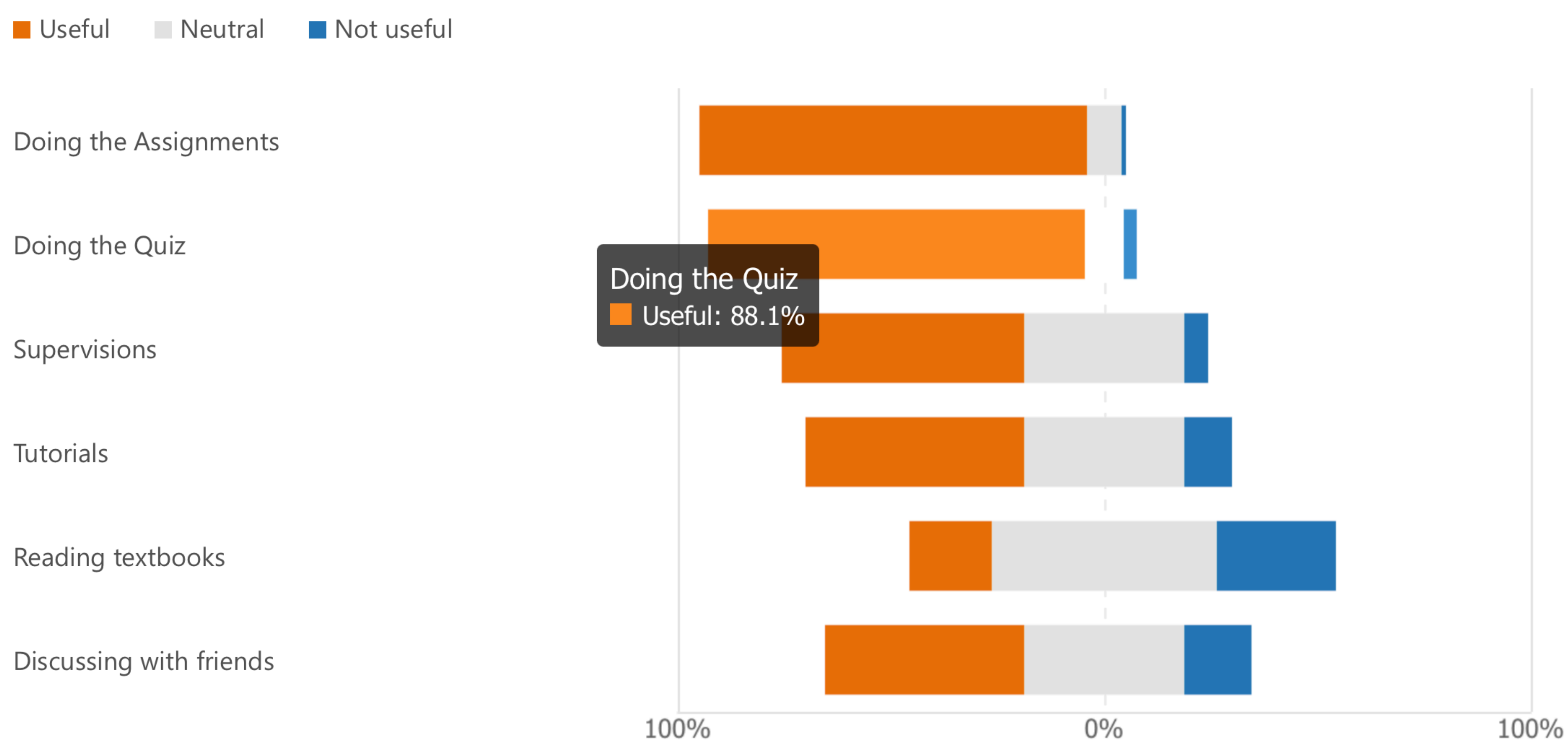


% of students who passed N quizzes
(Green sector not the largest)

Feedback

Problem sheet vs online quiz

1. Apart from my lecture notes and videos, rate the following in terms of usefulness for your understanding of Analysis II?



Written feedback

Name one thing about the module which has had the most impact on your learning.

The quizzes were really helpful learning resources. It really helped them being untimed, and the attempts being unlimited.

The quizzes were helpful, especially the fact that they were not timed and could be reattempted an infinite number of times insured that I did not feel any sort of pressure whilst doing them - allowing me to focus on understanding the content.

The quiz is really good at helping you check your understanding at the end of each week.

The quizzes have been a very useful interactive learning tool.

I liked the weekly quizzes that mixed content and examples.

The alternating assessment format between quizzes and written assignments is nice and helped reduce stress.

Name one way in which the module could be improved.

I would prefer if the quizzes did not have core content in them. This is because, if all content was covered in the workbooks it would be much nicer to work through/revise. I think the quizzes are better suited for practising.

The quizzes could be made more difficult - I'll admit that with quite a few of them it was easy to use exam technique and tricks to guess correct answers without properly thinking or taking in the content.

Sometimes quizzes are too guided.

Lessons learnt

- I saved a lot of lecture time by putting proofs in the Quiz. Students understand the proofs much better.
- Students appreciated practice quizzes in my module. Everyone hates timed quizzes.
- Duplicate quizzes so students can practise post-deadline.

Next step: Geometry & Motion (21/22)

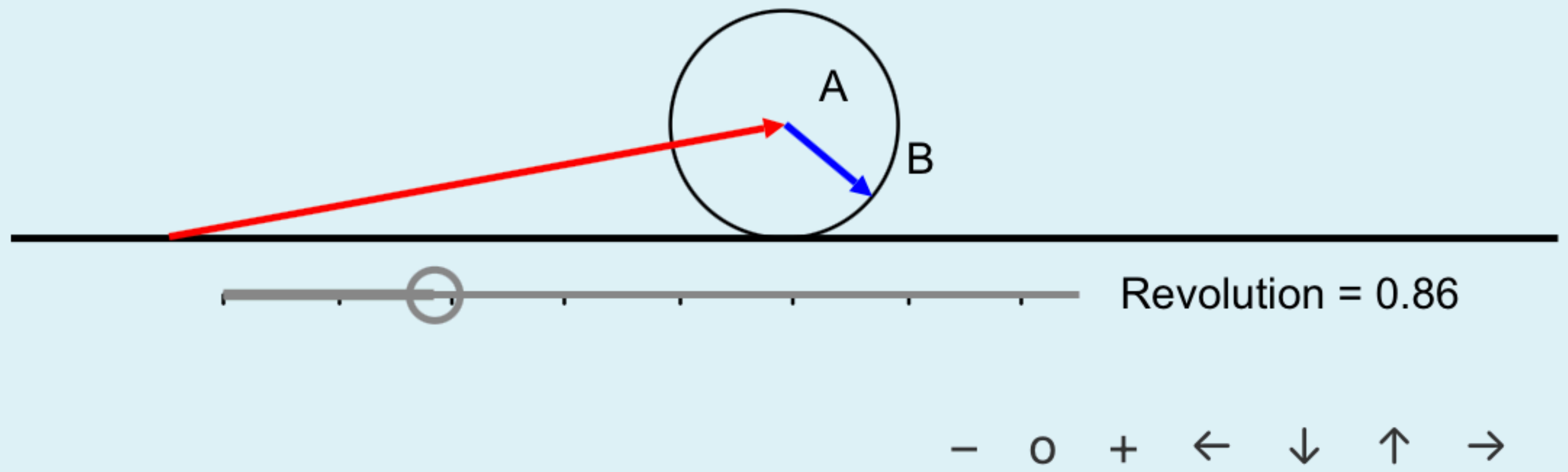
Intern: Wei Xiang Tan

On the figure, we have drawn in 2 vectors:

Vector **A** (red) is the position vector of the centre of the wheel

Vector **B** (blue) points radially from the centre of the wheel to the reference point.

Answer the following questions in terms of the velocity v and time variable t.

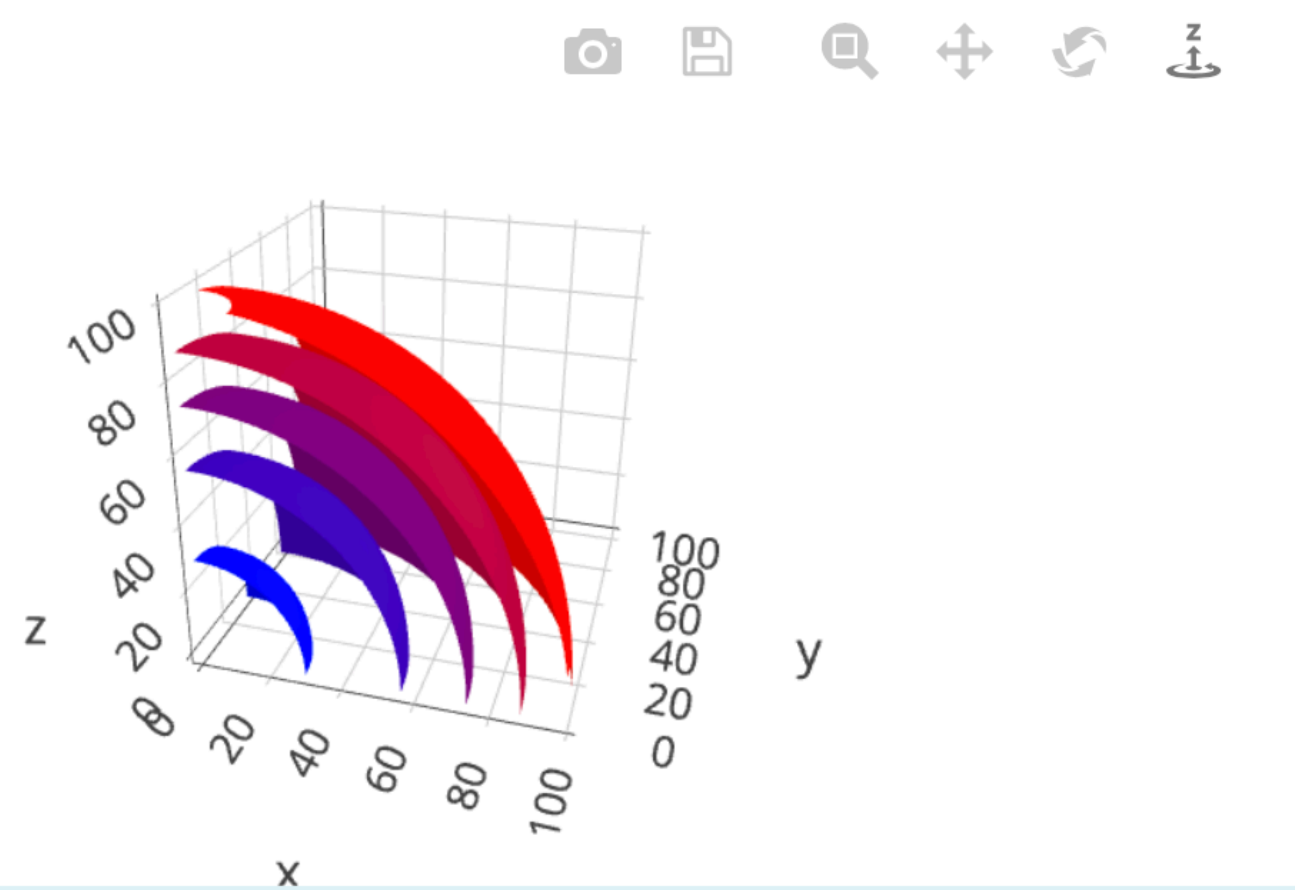


a) How long does it take for the wheel to complete 1 revolution?

Roll the wheel (derive cycloid eq.)

The interactive figure below shows surfaces given by

$$T(x, y, z) = x^2 + y^2 + z^2 = \text{constant.}$$

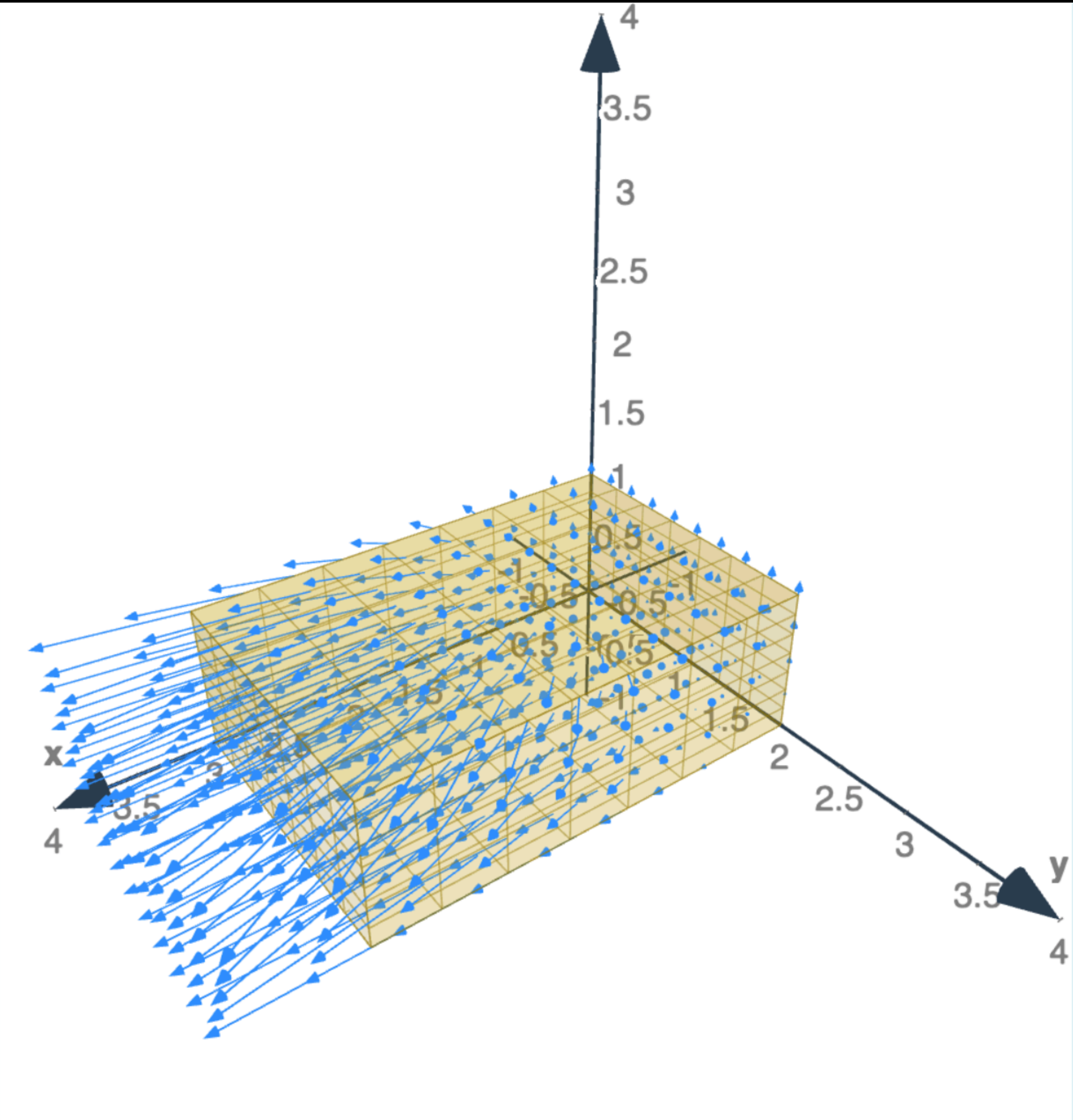


$T(x, y, z)$ represents the temperature at the point (x, y, z) in unit of Kelvin. These surfaces are called *isosurfaces* or *level sets*. The surfaces go from "cold" (blue, near the origin) to "hot" (red, further from the origin)

a) Give the coordinates of a point with temperature 100 Kelvin.

Ans: (, ,).

Interactive 3D plots (isosurfaces and grad)



Consider the vector field

$$\mathbf{F}(x, y, z) = (2x^2, 3xyz, 7z),$$

defined in $\mathbb{R}^3$. We wish to calculate the flux of $\mathbf{F}$ out of the cuboid

$$V = \{0 \leq x \leq 3,\ 0 \leq y \leq 2,\ 0 \leq z \leq 1\}.$$

Interactive 3D plots (verify divergence theorem)

How to get started with STACK?

- Log on to https://moodlex.warwick.ac.uk and send me an email. I'll register you on my "sandbox" page where you can play with STACK.
- Easy to follow guides on YouTube https://www.youtube.com/channel/UCkdewa3GAHr-OCA0QVjd3Ew
- Recruit a student or two as summer interns to help coding (IATL or WIHEA can fund co-creation projects)
- Coding and testing can take months - start early!

Conclusions

- E-assessments have come a long way from MCQs! You can now turn proofs, figures, visualisations into interactive e-assessments.
- STACK and other Moodle-type questions can be used in conjunction to increase engagement in your module.
- Feedback: students find quizzes as useful as doing traditional example sheets.
- Students can help create/test questions - it benefits everyone. Find an intern next summer! (write to me and we can co-apply)
- Start work early. Not easy, but reward is long term.