Determination of modular forms by fundamental Fourier coefficients

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Setting

- $V =$ some set of "modular forms".
- $\mathcal{S} =$ a set that indexes "Fourier coefficients" of elements of V, i.e., for all $\Phi \in V$, have an expansion

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- $\mathcal{D}$ = an "interesting subset" of $\mathcal{S}$.

We are interested in situations where the following implication is true for all $\Phi \in V$:

$$\Phi_n = 0 \ \forall \ n \in \mathcal{D} \qquad \Rightarrow \qquad \Phi = 0$$

or, equivalently:

$$\Phi \neq 0 \qquad \Rightarrow \qquad \text{there exists } n \in \mathcal{D} \text{ such that } \Phi_n \neq 0.$$

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This talk will focus on the following types of modular forms:

1. Modular forms of half-integral weight (automorphic forms on $\widetilde{\mathrm{SL}_2}$)
2. Siegel modular forms of degree 2 and trivial central character (automorphic forms on PGSp_4)

Definition of Sp_4

For a commutative ring R, we denote by $\mathrm{Sp}_4(R)$ the set of 4×4 matrices A satisfying the equation $A^t J A = J$ where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$.

Definition of $\mathbb{H}_2$

Let $\mathbb{H}_2$ denote the set of 2×2 matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

$\mathbb{H}_2$ is a homogeneous space for $\mathrm{Sp}_4(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

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The congruence subgroup $\Gamma_0^{(2)}(N)$

Let $\Gamma_0^{(2)}(N) \subset \mathrm{Sp}_4(\mathbb{Z})$ denote the subgroup of matrices that are congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod N.

The space $S_k(\Gamma_0^{(2)}(N))$

Siegel modular forms

A Siegel modular form of degree 2, level N, trivial character and weight k is a holomorphic function F on $\mathbb{H}_2$ satisfying

$$F(\gamma Z) = \det(CZ + D)^k F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N)$,
If in addition, F vanishes at the cusps, then F is called a cusp form.

We define $S_k(\Gamma_0^{(2)}(N))$ to be the space of cusp forms as above.

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Remark. As in the classical case, we have Hecke operators and a Petersson inner product.

Remark. Hecke eigenforms in $S_k(\Gamma_0^{(2)}(N))$ give rise to cuspidal automorphic representations of $\mathrm{PGSp}_4(\mathbb{A})$

Let $F(Z) \in S_k(\Gamma_0^{(2)}(N))$. Note that

$$F(Z + \begin{pmatrix} p & q \\ q & r \end{pmatrix}) = F(Z), \qquad \text{for all } Z \in \mathbb{H}_2, \ (p,q,r) \in \mathbb{Z}^3$$

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The Fourier expansion

$$F(Z) = \sum_{S>0} a(F,S) e^{2\pi i \operatorname{Tr} SZ}$$

where S varies over all matrices $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $(a,b,c) \in \mathbb{Z}^3$ and $b^2 < 4ac$. We denote $\operatorname{disc}(S) = b^2 - 4ac$.

Remark. The Fourier coefficients $a(F,S)$ are mysterious objects and are conjecturally related to central L-values (when F is an eigenform).

Fourier coefficients with fundamental discriminant

Recall the Fourier expansion $F(Z) = \sum_{S>0} a(F,S) e^{2\pi i \mathrm{Tr} SZ}$.

Note that $\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \in \Gamma_0^{(2)}(N)$ for all $A \in \mathrm{SL}_2(\mathbb{Z})$.

$\mathrm{SL}_2(\mathbb{Z})$-invariance of Fourier coefficients

This shows that

$$a(F, ASA^t) = a(F,S)$$

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We are interested in situations where F is determined by the Fourier coefficients $a(F,S)$ with $\mathrm{disc}(S) < 0$ a fundamental discriminant.

Recall: $d \in \mathbb{Z}$ is a fundamental discriminant if EITHER d is a squarefree integer congruent to 1 mod 4 OR $d = 4m$ where m is a squarefree integer congruent to 2 or 3 mod 4.

The main result

The $U(p)$ operator

For all $p|N$, we have an operator $U(p)$ on $S_k(\Gamma_0^{(2)}(N))$ defined by

$$(U(p)F)(Z) = \sum_{S>0} a(F, pS) e^{2\pi i \mathrm{Tr} SZ}.$$

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Theorem 1 (S – Schmidt)

Let N be squarefree. Let $k > 2$ be an integer, and if $N > 1$ assume k even. Let $F \in S_k(\Gamma_0^{(2)}(N))$ be non-zero and an eigenfunction of the $U(p)$ operator for all $p|N$. Then $a(F, S) \neq 0$ for infinitely many S with $\mathrm{disc}(S)$ a fundamental discriminant.

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Remark. If $N = 1$, no $U(p)$ condition.

Remark. In fact we can give the lower bound $X^{\frac{5}{8}-\epsilon}$ for the number of such non-vanishing Fourier coefficients with absolute discriminant less than X.

- $V =$ the elements of $S_k(\Gamma_0^{(2)}(N))$ that are eigenfunctions of $U(p)$ for $p|N$.
- $\mathcal{S} =$ the set of matrices $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $(a, b, c) \in \mathbb{Z}^3$ and $b^2 < 4ac$. For all $\Phi \in V$, we have a Fourier expansion

$$\Phi(Z) = \sum_{n \in \mathcal{S}} \Phi_n(Z).$$

- $\mathcal{D} =$ the subset of $\mathcal{S}$ consisting of those matrices with $b^2 - 4ac$ a fundamental discriminant.

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Theorem 1 says: For all $\Phi \in V$,

$$\Phi_n = 0 \ \forall \ n \in \mathcal{D} \qquad \Rightarrow \qquad \Phi = 0$$

or, equivalently:

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Theorem 1

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Why do we care?

Key point: From the automorphic point of view, weighted averages of Fourier coefficients of Siegel modular forms are simultaneously

- Period integrals over Bessel subgroups
- (Conjecturally) Central L-values of quadratic twists of the relevant automorphic representation

As a result, non-vanishing of Fourier coefficients leads to very interesting consequences.

Why do we care? (contd.)

Let $F(Z) \in S_k(\Gamma_0^{(2)}(N))$ be a Hecke eigenform. Let $-d < 0$ be a fundamental discriminant and put $K = \mathbb{Q}(\sqrt{-d})$. Let Cl_K denote the ideal class group of K.

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The following fact goes back to Gauss:

$$\mathrm{SL}_2(\mathbb{Z}) \setminus \left\{ S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \ \mathrm{disc}(S) = -d \right\} \quad \cong \quad \mathrm{Cl}_K .$$

Recall that $a(F, ASA^t) = a(F, S)$ for all $A \in \mathrm{SL}_2(\mathbb{Z})$

So, for any character Λ of the finite group Cl_K, the following quantity is well-defined,

$$R(F, d, \Lambda) = \sum_{c \in \mathrm{Cl}_K} a(F, c) \Lambda^{-1}(c)$$

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Corollary of Theorem 1

There are infinitely many d, Λ as above, so that $R(F, d, \Lambda) \neq 0$.

Bessel models

The automorphic representation Π_F of PGSp_4 attached to F does not have a Whittaker model. So many automorphic methods that rely on Whittaker models do not work. However the non-vanishing of $R(F, d, \Lambda)$ means that it has a Bessel model of a very nice type!

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In a pioneering paper, Furusawa (1993) proved an integral representation and special value results for GL_2 twists of Π_F having such a nice Bessel model. Several subsequent papers by Pitale-Schmidt (2009), S (2009, 2010) and Pitale–S–Schmidt (2011) proved results for Π_F under the same assumption.

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With Theorem 1, we now know that all those results hold unconditionally for Π_F coming from eigenforms in $S_k(\Gamma_0^{(2)}(N))$.

Central L-values

We continue to assume that $F(Z) \in S_k(\Gamma_0^{(2)}(N))$ is a eigenform, $-d < 0$ a fundamental discriminant, Λ an ideal class character of $K = \mathbb{Q}(\sqrt{-d})$ and $R(F, d, \Lambda) = \sum_{c \in \mathrm{Cl}_K} a(F, c)\Lambda^{-1}(c)$.

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A generalization of a conjecture of Böcherer by several people (Böcherer, Furusawa, Shalika, Martin, Prasad, Takloo-Bighash), leads to the following very interesting Gross-Prasad type conjecture.

Conjecture

Suppose for some F, d, Λ as above, we have $R(F, d, \Lambda) \neq 0$. Then $L(\frac{1}{2}, \Pi_F \times \theta_\Lambda) \neq 0$, where $\theta_\Lambda = \sum_{0 \neq a \subset O_K} \Lambda(a) e^{2\pi i N(a) z}$ is a holomorphic modular form of weight 1 and nebentypus $(\frac{-d}{*})$ on $\Gamma_0(d)$.

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The above conjecture is not proved in general; however it is known for certain special Siegel cusp forms that are *lifts*.

Yoshida lifts

- N_1, N_2 : two squarefree integers that are not coprime.
- $N = \text{lcm}(N_1, N_2)$.
- f : newform of weight 2 on $\Gamma_0(N_1)$.
- g : newform of weight $2k$ on $\Gamma_0(N_2)$.
- Assume that for all $p | \gcd(N_1, N_2)$, f and g have the same Atkin-Lehner eigenvalue at p.

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The Yoshida lift

Under the above assumptions, there exists a eigenform $F \in S_{k+1}(\Gamma_0^{(2)}(N))$ such that

$$L(s, \Pi_F) = L(s, \pi_f) L(s, \pi_g)$$

Remark. In the language of automorphic representations, the Yoshida lift is a special case of Langlands functoriality, coming from the embedding of L-groups

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C}).$$

How is the Yoshida lift constructed?

The Yoshida lift is constructed via the theta correspondence. Suppose we start with classical newforms f, g as in the previous slide.

1. First we fix a definite quaternion algebra D which is unramified at all finite primes outside $\gcd(N_1, N_2)$.
2. Via the Jacquet-Langlands correspondence, we transfer π_f, π_g to representations π'_f, π'_g on $D^\times(\mathbb{A})$.
3. Using the isomorphism
$$(D^\times \times D^\times)/\mathbb{Q}^\times \cong GSO(4)$$
we obtain an automorphic representation $\pi'_{f,g}$ on $GSO(4, \mathbb{A})$.
4. Finally we use the theta lifting to transfer $\pi'_{f,g}$ to the automorphic representation Π_F on $\mathrm{GSp}_4(\mathbb{A})$.

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Recall the conjecture stated earlier which is expected to hold for any Siegel eigenform F, a fundamental discriminant $-d$ and an ideal class character Λ of $\mathbb{Q}(\sqrt{-d})$.

Conjecture

Suppose we have $R(F, d, \Lambda) \neq 0$. Then $L(\frac{1}{2}, \Pi_F \times \theta_\Lambda) \neq 0$.

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Theorem (Prasad–Takloo-Bighash)

The above conjecture is true when F is a Yoshida lifting.

Remark. If $\Lambda = 1$, this is also proved in work of Böcherer–Schulze-Pillot.
Remark. Note that when F is a Yoshida lifting, then

$$L(\frac{1}{2}, \Pi_F \times \theta_\Lambda) = L(\frac{1}{2}, \pi_f \times \theta_\Lambda) L(\frac{1}{2}, \pi_g \times \theta_\Lambda).$$

What we have so far

Yoshida lift

Given f , g classical newforms satisfying some compatibility conditions, there exists a eigenform $F \in S_{k+1}(\Gamma_0^{(2)}(N))$ such that $L(s, \Pi_F) = L(s, \pi_f)L(s, \pi_g)$

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Corollary of Theorem 1

We can find infinitely many pairs (d, Λ) with $-d$ a fundamental discriminant and Λ an ideal class group character of $\mathbb{Q}(\sqrt{-d})$ such that $R(F, d, \Lambda) \neq 0$.

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Theorem of Prasad–Takloo-Bighash

$$R(F, d, \Lambda) \neq 0 \quad \Rightarrow \quad L(\frac{1}{2}, \pi_f \times \theta_\Lambda)L(\frac{1}{2}, \pi_g \times \theta_\Lambda) \neq 0.$$

A simultaneous non-vanishing result

Putting together the three results of the previous slide, we obtain the following result:

Theorem 2 (S–Schmidt)

Let $k > 1$ be an odd positive integer. Let N_1, N_2 be two positive, squarefree integers such that $M = \gcd(N_1, N_2) > 1$. Let f be a holomorphic newform of weight $2k$ on $\Gamma_0(N_1)$ and g be a holomorphic newform of weight 2 on $\Gamma_0(N_2)$. Assume that for all primes p dividing M the Atkin-Lehner eigenvalues of f and g coincide. Then there exists an imaginary quadratic field K and a character $\chi \in \widehat{\mathrm{Cl}_K}$ such that $L(\frac{1}{2}, \pi_f \times \theta_\chi) \neq 0$ and $L(\frac{1}{2}, \pi_g \times \theta_\chi) \neq 0$.

Remark. Our proof shows, in fact, that there are at least $X^{\frac{5}{8}-\epsilon}$ such pairs (K, χ) with $\mathrm{disc}(K) < X$.

Thus we have seen that Theorem 1 leads to

- Existence of nice Bessel models for automorphic representations attached to Siegel eigenforms. This makes several old results of Furusawa, Pitale, Saha, Schmidt unconditional.
- Simultaneous non-vanishing of dihedral twists of two modular L-functions.

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How is Theorem 1 proved?

It turns out that the key step of proving Theorem 1 is a very similar result for modular forms of half-integral weight!

Classical modular forms of half-integral weight

Let N be a squarefree integer. For any non-negative integer k, let $S_{k+\frac{1}{2}}(\Gamma_0(4N))$ denote the space of cusp forms of weight $k+\frac{1}{2}$, level $4N$ and trivial character.

Let $S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ denote the Kohnen subspace of $S_{k+\frac{1}{2}}(\Gamma_0(4N))$.

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Fourier expansion

Any $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ has a Fourier expansion

$$f(z) = \sum_{(-1)^k n \equiv 0,1(4)} a(f,n) e^{2\pi i z}.$$

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$$f(z) = \sum_{(-1)^k n \equiv 0,1(4)} a(f,n)e^{2\pi i z}.$$

Let $\mathcal{D}$ be the set of integers $d > 0$ such that $(-1)^k d$ is a fundamental discriminant.

Remark. If $f \in S^+_{k+1/2}(\Gamma_0(4N))$ is a newform, then Waldspurger's theorem (worked out precisely in this case by Kohen) implies that $|a(f,d)|^2$ is essentially equal to $L(1/2, \pi \times \chi_d)$.

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We are interested in the situation when elements of $S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ *are determined by the Fourier coefficients* $a(f,d)$ *with* $d \in \mathcal{D}$.

In the language of our setup,

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Theorem 3 (S)

Let $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ where N is squarefree and $k \geq 2$. Assume $f \neq 0$. Then $a(f, d) \neq 0$ for infinitely many d in $\mathcal{D}$.

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Remark. Note that because f is not a Hecke eigenform, there is no way to reduce the problem to central L-values!

Remark. Actually the theorem I prove is stronger: N can be divisible by squares of primes, the nebentypus need not be trivial, one can work with the larger space $S_{k+\frac{1}{2}}(\Gamma_0(4N))$, and one can give a lower bound on the number of non-vanishing Fourier coefficients $a(f, d)$.

A quick recap of the two results

Theorem 1

Let N be squarefree. Let $k > 2$ be an integer, and if $N > 1$ assume k even. Let $F \in S_k(\Gamma_0^{(2)}(N))$ be non-zero and an eigenfunction of the $U(p)$ operator for all $p|N$. Then $a(F, S) \neq 0$ for infinitely many S with $\mathrm{disc}(S)$ a fundamental discriminant.

Theorem 3

Let $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ where N is squarefree and $k \geq 2$. Assume $f \neq 0$. Then $a(f, d) \neq 0$ for infinitely many d in $\mathcal{D}$.

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Rest of this talk:

- why Theorem 3 implies Theorem 1.
- how Theorem 3 is proved.

Why does Theorem 3 imply Theorem 1?

For simplicity, let us restrict to the case $N = 1$ and k even.

- Let $F(Z) = \sum_S a(F,S) e^{2\pi i \mathrm{Tr} SZ} \in S_k(\Gamma_0^{(2)}(1)), \;\; F \neq 0.$
- Need to find $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ such that $b^2 - 4ac$ is a fundamental discriminant and $a(F,S) \neq 0$.

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Step 1. Using a result of Zagier, one can show that there exist a matrix $S' = \begin{pmatrix} a & b/2 \\ b/2 & p \end{pmatrix}$ such that $a(F,S') \neq 0$ and p is an odd *prime* number.

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Step 2. For each $n \geq 1$, define

$$c(n) = a\left(F, \begin{pmatrix} \frac{n+b^2}{4p} & b/2 \\ b/2 & p \end{pmatrix}\right)$$

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$$h(z) = \sum_{n \geq 1} c(n)e^{2\pi i n z}.$$

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Because of Step 1, it follows that $h(z) = \sum_{n \geq 1} c(n) e^{2\pi i n z} \neq 0$.

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Theorem (Eichler–Zagier, Skoruppa)

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Step 3. It follows from Theorem 3 that $c(d) \neq 0$ for infinitely many d such that $-d$ is a fundamental discriminant. Since

$$c(d) = a\left(F, \begin{pmatrix} \frac{d+b^2}{4p} & b/2 \\ b/2 & p \end{pmatrix}\right)$$

this proves Theorem 1.

How to prove Theorem 3?

- Let $f \in S^{+}_{k+\frac{1}{2}}(\Gamma_0(4N))$ be non-zero and $\mathcal{D}$ be the set of integers d such that $(-1)^k d$ is a fundamental discriminant. Need to show there exists $d \in \mathcal{D}$ with $a(f, d) \neq 0$.

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- The key point is to consider the following quantity for any integer M, and any $X > 0$,

$$S(M, X; f) = \sum_{\substack{d \in \mathcal{D} \\ (d,M)=1}} d^{\frac{1}{2}-k} |a(f, d)|^2 e^{-d/X}.$$

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- We prove that there exists an integer M and a constant $C_f > 0$ so that $S(M,X;f) > C_f X$ for all sufficiently large X.
- This shows that there are infinitely many $d \in \mathcal{D}$ such that $a(f,d) \neq 0$.