Introduction to Chevalley Groups

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Definition

A *Lie group* is a smooth manifold G equipped with a group structure so that the maps $\mu : (x, y) \mapsto xy$, $G \times G \to G$ and $\iota : x \mapsto x^{-1}$, $G \to G$ are smooth.

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Definition

A *Lie algebra* is a vector space $\mathfrak{L}$ over a field K on which a product operation $[x, y]$ is defined satisfying the following axioms:

1. $[x, y]$ is bilinear for all $x, y \in \mathfrak{L}$.
2. $[x, x] = 0$ for all $x \in \mathfrak{L}$.
3. (Jacobi identity) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for $x, y, z \in \mathfrak{L}$.

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This is a one-to-one correspondence between connected simple Lie groups with trivial centre and simple Lie algebras.

Definition

Let V be a finite dimensional Euclidean space. For each non-zero vector r of V we denote by w_r the reflection in the hyperplane orthogonal to r. If x is any vector in V, this reflection is given by

$$w_r(x) = x - \frac{2(r,x)}{(r,r)}r.$$

A subset Φ of V is called a *root system* in V if the following hold:

1. Φ is a finite set of non-zero vectors.
2. Φ spans V.
3. If $r, s \in \Phi$ then $w_r(s) \in \Phi$.
4. If $r, s \in \Phi$ then $\frac{2(r,s)}{(r,r)}$ is an integer.
5. If $r, \lambda r \in \Phi$, where $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

The elements of Φ are called *roots*.

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Every choice of Π determines two subsets of Φ:

- a subset of *positive roots*, denoted by Φ^+ (all of whose coefficients are non-negative)
- a subset of *negative roots*, denoted by Φ^- (all of whose coefficients are non-positive).

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Each element of W transforms Φ into itself, and W operates faithfully on Φ. Since Φ is a finite set, W is a finite group.

Let J be a subset of Π. We define V_J to be the subspace of V spanned by J; Φ_J to be $\Phi \cap V_J$; and W_J to be the subgroup of W generated by the reflections w_r with $r \in J$.

Proposition

Φ_J is a root system in V_J. J is a fundamental system in Φ_J. The Weyl group of Φ_J is W_J.

Definition

The subgroups W_J and their conjugates in W are called *parabolic subgroups* of W.

Proposition

The subgroups W_J for distinct subsets J of Π are all distinct.

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Theorem

Let J, K be subsets of Π. Then

1. *the subgroup of W generated by W_J and W_K is $W_{J \cup K}$*
2. $W_J \cap W_K = W_{J \cap K}$.

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So the parabolic subgroups W_J form a lattice in W that is in bijection with the lattice of subsets of Π.

Definition

For each element x of a Lie algebra $\mathfrak{L}$ we define a linear map $\mathrm{ad}\, x : \mathfrak{L} \to \mathfrak{L}$ by

$$\mathrm{ad}\, x.y = [x, y] \qquad y \in \mathfrak{L}.$$

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$\mathrm{ad}\, x$ is also a *derivation* of $\mathfrak{L}$, meaning that it satisfies the Leibniz rule:

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For each $x, y \in \mathfrak{L}$ we define the *Killing form* (x, y) by

$$(x, y) = \text{tr } (\text{ad } x \ . \ \text{ad } y).$$

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Any associative algebra can be turned into a Lie algebra by defining the Lie product as $[x, y] = xy - yx$. So the algebra of $n \times n$ matrices is an example of a Lie algebra.

Definition

A subalgebra $\mathfrak{H}$ of a Lie algebra $\mathfrak{L}$ is called a Cartan subalgebra if it satisfies:

1. $\mathfrak{H}$ is nilpotent, i.e. there exists an r such that $\underbrace{[[[\mathfrak{H},\mathfrak{H}],\mathfrak{H}]\cdots]}_{r}=0$.
2. $\mathfrak{H}$ is self-normalising, i.e. it is not contained as an ideal in any larger subalgebra of $\mathfrak{L}$.

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Every Lie algebra over $\mathbb{C}$ has a Cartan subalgebra, and any two Cartan subalgebras are isomorphic. The dimension of the Cartan subalgebras is called the *rank* of $\mathfrak{L}$, usually denoted by l.

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A Lie algebra is said to be *simple* if it has no ideals other than itself and the zero subspace. For a simple Lie algebra over $\mathbb{C}$ we have $[\mathfrak{H}, \mathfrak{H}] = 0$.

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{L}$. Then $\mathfrak{L}$ can be decomposed into a direct sum as follows:

$$\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{L}_{r_1} \oplus \mathfrak{L}_{r_2} \oplus \cdots \oplus \mathfrak{L}_{r_k}$$

where

- each $\mathfrak{L}_{r_i}$ has dimension 1
- each $\mathfrak{L}_{r_i}$ is invariant under Lie multiplication by $\mathfrak{H}$, i.e. $[\mathfrak{H}, \mathfrak{L}_{r_i}] = \mathfrak{L}_{r_i}$ for each i.

This is called a *Cartan decomposition* of $\mathfrak{L}$.

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A Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{sl}_3$ is given by the diagonal matrices. We can check that we indeed have $[\mathfrak{H}, \mathfrak{H}] = 0$:

$$\left[\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}\right] = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix} - \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$
$$= \begin{pmatrix} ad - da & 0 & 0 \\ 0 & be - eb & 0 \\ 0 & 0 & cf - fc \end{pmatrix} = 0.$$

This subalgebra has dimension 2.

The subspaces $\mathfrak{L}_{r_i}$ are the 1-dimensional subspaces spanned by the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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We can check that these subspaces are indeed invariant under multiplication by elements of $\mathfrak{H}$:

$$\begin{aligned}\left[\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right] &= \begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\\ &= \begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}0&b&0\\0&0&0\\0&0&0\end{pmatrix}\\ &= \begin{pmatrix}0&a-b&0\\0&0&0\\0&0&0\end{pmatrix}.\end{aligned}$$

There are 6 of these subspaces, so $\mathfrak{sl}_3$ has dimension 8.

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{L}_{r_1} \oplus \mathfrak{L}_{r_2} \oplus \cdots \oplus \mathfrak{L}_{r_k}$ be a Cartan decomposition.

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In each 1-dimensional subspace $\mathfrak{L}_r$ we pick a non-zero element e_r. Then for each $h \in \mathfrak{H}$, $[he_r]$ is a scalar multiple of e_r and we write

$$[he_r] = r(h)\, e_r .$$

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The map $r : \mathfrak{H} \to \mathbb{C}$ defined like this is linear, so it is an element of $\mathfrak{H}^*$.

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Definition

The maps $r_1, r_2, \ldots, r_k$ of $\mathfrak{H}$ to $\mathbb{C}$ are called the *roots* of $\mathfrak{L}$ and the subspaces $\mathfrak{L}_{r_1}, \mathfrak{L}_{r_2}, \ldots, \mathfrak{L}_{r_k}$ are called the *root spaces* of $\mathfrak{L}$ (relative to the given Cartan subalgebra $\mathfrak{H}$).

The roots $r_1, r_2, \ldots, r_k$ are all distinct and non-zero.

The roots are defined as elements of $\mathfrak{H}^*$, but we can also view them as elements of $\mathfrak{H}$ as follows.

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The Killing form of a simple Lie algebra $\mathfrak{L}$ is non-singular. So it remains non-singular when restricted to $\mathfrak{H}$. So each element of $\mathfrak{H}^*$ is expressible in the form $h \mapsto (x, h)$ for a unique element $x \in \mathfrak{H}$.

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The element x associated to the map $h \mapsto r(h)$ may be identified with the root r. So r can be regarded either as an element of $\mathfrak{H}$ or as an element of $\mathfrak{H}^*$, the relationship between these two being:

$$r(h) = (r, h), \qquad h \in \mathfrak{H}.$$

Let Φ denote the finite set of roots viewed as a subset of $\mathfrak{H}$.

Let $\mathfrak{H}_{\mathbb{R}}$ denote the set of all $\mathbb{R}$-linear combinations of Φ. Then $\mathfrak{H}_{\mathbb{R}}$ is a real vector space of the same dimension as the complex dimension of $\mathfrak{H}$. The Killing form is positive definite on $\mathfrak{H}_{\mathbb{R}}$, so $\mathfrak{H}_{\mathbb{R}}$ can be regarded as a Euclidean space.

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Then the set of roots Φ form a root system as defined previously.

Suppose that r, s are linearly independent roots. Since the set Φ is finite, the sequence of roots

$$-pr + s, \ldots, s, \ldots, qr + s$$

(for $p, q \geq 0$) is finite. This is called the *r-chain of roots through s*.

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For example, in the root system B_2, the a-chain of roots through $-2a - b$ is

$$-2a - b, \ -a - b, \ -b$$

so in this case $p = 0$ and $q = 2$.

The reflection w_r acts on the root s by $w_r(s) = s - \frac{2(r,s)}{(r,r)} r$. In fact w_r has the effect of inverting each r-chain of roots. So $-pr + s$ and $qr + s$ are mirror images in the hyperplane orthogonal to r. So

$$((-pr + s) + (qr + s), r) = 0.$$

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It follows that $\frac{2(r,s)}{(r,r)} = p - q$, so if we define

$$A_{rs} = \frac{2(r,s)}{(r,r)}$$

then A_{rs} is an integer which satisfies $w_r(s) = s - A_{rs}r$ and $A_{rs} = p - q$.

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If we take the r, s to be fundamental roots, then the integers A_{rs} are the entries of the *Cartan matrix* of $\mathfrak{L}$.

Theorem

Let Φ be an indecomposable root system. Then there exists a simple Lie algebra over $\mathbb{C}$ which has a root system equivalent to Φ.

Theorem

Let $\mathfrak{L}$, $\mathfrak{L}'$ be simple Lie algebras over $\mathbb{C}$ with Cartan subalgebras $\mathfrak{H}$, $\mathfrak{H}'$ of the same dimension l. Let $p_1, p_2, \ldots, p_l$ and $p'_1, p'_2, \ldots, p'_l$ be sets of fundamental roots for $\mathfrak{L}$, $\mathfrak{L}'$ and let

$$A_{ij} = \frac{2(p_i, p_j)}{(p_i, p_i)}, \quad A'_{ij} = \frac{2(p'_i, p'_j)}{(p'_i, p'_i)}.$$

Let

$$h_{p_i} = \frac{2p_i}{(p_i, p_i)}$$

and let $e_{p_i} \in \mathfrak{L}_{p_i}$, $e_{-p_i} \in \mathfrak{L}_{-p_i}$ be chosen so that $[e_{p_i}, e_{-p_i}] = h_{p_i}$. Define $h_{p'_i}, e_{p'_i}, e_{-p'_i}$ similarly in $\mathfrak{L}'$.
Suppose $A_{ij} = A'_{ij}$ for all i, j. Then there exists a unique isomorphism $\theta : \mathfrak{L} \to \mathfrak{L}'$ such that $\theta(h_{p_i}) = h_{p'_i}$, $\theta(e_{p_i}) = e_{p'_i}$, $\theta(e_{-p_i}) = e_{-p'_i}$.
In particular any two simple Lie algebras over $\mathbb{C}$ with equivalent root systems are isomorphic.

Any complex simple Lie algebra is isomorphic to one of the following:
A_l $(l \geq 1)$, of dimension $l(l+2)$
B_l $(l \geq 2)$, of dimension $l(2l+1)$
C_l $(l \geq 3)$, of dimension $l(2l+1)$
D_l $(l \geq 4)$, of dimension $l(2l-1)$
G_2, of dimension 14
F_4, of dimension 52
E_6, of dimension 78
E_7, of dimension 133
E_8, of dimension 248

Theorem

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ and

$$\mathfrak{L} = \mathfrak{H} \oplus \sum_{r \in \Phi} \mathfrak{L}_r$$

be a Cartan decomposition of $\mathfrak{L}$. Let $h_r \in \mathfrak{L}_r$ be the co-root corresponding to the root r. Then, for each root $r \in \Phi$, an element e_r can be chosen in $\mathfrak{L}_r$ such that

$$[e_r, e_{-r}] = h_r,$$
$$[e_r, e_s] = \pm(p+1)e_{r+s},$$

where p is the greatest integer for which $s - pr \in \Phi$.
The elements $\{h_r, r \in \Pi; e_r, r \in \Phi\}$ form a basis for $\mathfrak{L}$, called a Chevalley basis. The basis elements multiply together as follows:

$$\begin{aligned}
[h_r, h_s] &= 0, \\
[h_r, e_s] &= A_{rs} e_s, \\
[e_r, e_{-r}] &= h_r, \\
[e_r, e_s] &= 0 \qquad \text{if } r + s \notin \Phi, \\
[e_r, e_s] &= N_{r,s} e_{r+s} \qquad \text{if } r + s \in \Phi,
\end{aligned}$$

where $N_{r,s} = \pm(p+1)$.
The multiplication constants of the algebra with respect to the Chevalley basis are all integers.

Lemma

Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic 0 and δ be a derivation of $\mathfrak{L}$ which is nilpotent, i.e. satisfies $\delta^n = 0$ for some n. Then

$$\exp \delta = 1 + \delta + \frac{\delta^2}{2} + \cdots + \frac{\delta^{n-1}}{(n-1)!}$$

is an automorphism of $\mathfrak{L}$.

Fact: if $\mathfrak{L}$ is a simple Lie algebra over $\mathbb{C}$ with Cartan decomposition $\mathfrak{L} = \mathfrak{H} \oplus \sum_{r\in\Phi} \mathfrak{L}_r$ and Chevalley basis $\{h_r, r \in \Pi; e_r, r \in \Phi\}$, then the map ad e_r is a nilpotent derivation of $\mathfrak{L}$.
Let $\zeta \in \mathbb{C}$. Then ad $(\zeta e_r) = \zeta$ad e_r is also a nilpotent derivation of $\mathfrak{L}$. So $\exp(\zeta \text{ ad } e_r)$ is an automorphism of $\mathfrak{L}$. We define

$$x_r(\zeta) = \exp(\zeta \text{ad } e_r).$$

$$x_r(\zeta).e_r = e_r,$$
$$x_r(\zeta) - e_{-r} = e_{-r} + \zeta h_r - \zeta^2 e_r \ ,$$
$$x_r(\zeta).h_r = h_r - 2\zeta e_r.$$

Also, if r and s are linearly independent:

$$x_r(\zeta).h_s = h_s - A_{sr}\zeta e_r,$$

$$x_r(\zeta).e_s = e_s + N_{r,s}\zeta e_{r+s} + \frac{1}{2!}N_{r,s}N_{r,r+s}\zeta^2 e_{2r+s} + \cdots + \frac{1}{q!}N_{r,s}N_{r,r+s}\cdots N_{r,(q-1)r+s}\zeta^q e_{qr+s}$$
$$= \sum_{i=0}^{q} M_{r,s,i}\zeta^i e_{ir+s},$$

where $M_{r,s,i} = \pm\binom{p+i}{i}$.

So the automorphism $x_r(\zeta)$ transforms each element of the Chevalley basis into a linear combination of basis elements, the coefficients being non-negative integral powers of ζ with rational integer coefficients.

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We define $\mathfrak{L}_{\mathbb{Z}}$ to be the subset of $\mathfrak{L}$ consisting of all $\mathbb{Z}$-linear combinations of basis elements. $\mathfrak{L}_{\mathbb{Z}}$ is then a Lie ring over $\mathbb{Z}$.

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Let K be any field. We form the tensor product of the additive group of K with the additive group of $\mathfrak{L}_{\mathbb{Z}}$ and define

$$\mathfrak{L}_K = K \otimes_{\mathbb{Z}} \mathfrak{L}_{\mathbb{Z}}.$$

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Then $\mathfrak{L}_K$ is a vector space over K with basis

$$\{1_K \otimes h_r, r \in \Pi; 1_K \otimes e_r, r \in \Phi\}$$

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ with Chevalley basis $\{h_r, r \in \Pi; e_r, r \in \Phi\}$.

We define $\mathfrak{L}_{\mathbb{Z}}$ to be the subset of $\mathfrak{L}$ consisting of all $\mathbb{Z}$-linear combinations of basis elements. $\mathfrak{L}_{\mathbb{Z}}$ is then a Lie ring over $\mathbb{Z}$.

Let K be any field. We form the tensor product of the additive group of K with the additive group of $\mathfrak{L}_{\mathbb{Z}}$ and define

$$\mathfrak{L}_K = K \otimes_{\mathbb{Z}} \mathfrak{L}_{\mathbb{Z}}.$$

Then $\mathfrak{L}_K$ is a vector space over K with basis

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The multiplication constants of $\mathfrak{L}_K$ with respect to this new basis are the multiplication constants of $\mathfrak{L}$ with respect to the old basis, interpreted as elements of the prime subfield of K.

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Let $t \in K$ and define $\bar{A}_r(t)$ to be the matrix obtained from $A_r(\zeta)$ by replacing each coefficient $a\zeta^i$ by $\bar{a}t^i$, where $\bar{a}$ is the element of the prime field of K corresponding to $a \in \mathbb{Z}$.

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Define $x_r(t)$ to be the linear map of $\mathfrak{L}_K$ into itself represented by $\bar{A}_r(t)$.

Then $x_r(t)$ is an automorphism of $\mathfrak{L}_K$ for each $r \in \Phi$, $t \in K$.

Definition

The (adjoint) *Chevalley group* of type $\mathfrak{L}$ over the field K, denoted by $\mathfrak{L}(K)$, is defined to be the group of automorphisms of the Lie algebra $\mathfrak{L}_K$ generated by the $x_r(t)$ for all $r \in \Phi$, $t \in K$.

Proposition

The group $\mathfrak{L}(K)$ is determined up to isomorphism by the simple Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ and the field K.

Theorem

Let K be any field.

1. $A_l(K)$ *is isomorphic to the linear group* $PSL_{l+1}(K)$.
2. $B_l(K)$ *is isomorphic to the orthogonal group* $P\Omega(K, f_B)$
3. $C_l(K)$ *is isomorphic to the symplectic group* $PSp_{2l}(K)$
4. $D_l(K)$ *is isomorphic to the orthogonal group* $P\Omega(K, f_D)$.

Fix $r \in \Phi$. Let X_r be the subgroup of $\mathfrak{L}(K)$ generated by the elements $x_r(t)$ for all $t \in K$.

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We have

$$\begin{aligned} x_r(t_1).x_r(t_2) &= \exp(t_1 \text{ ad } e_r).\exp(t_2 \text{ ad } e_r) \\ &= \exp((t_1 + t_2) \text{ ad } e_r) \\ &= x_r(t_1 + t_2). \end{aligned}$$

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Also, $x_r(t) = 1$ if and only if $t = 0$, so each root subgroup X_r is isomorphic to the additive group of the field K.

We define U to be the subgroup of $\mathfrak{L}(K)$ generated by the elements $x_r(t)$ for $r \in \Phi^+$, $t \in K$.

We define V to be the subgroup of $\mathfrak{L}(K)$ genetated by the elements $x_r(t)$ for $r \in \Phi^-$, $t \in K$.

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The subgroups U and V are called the *unipotent subgroups* because their elements operate on $\mathfrak{L}(K)$ as unipotent linear transformations (where a linear transformation u is said to be unipotent if $u - 1$ is nilpotent, or equivalently if all of the eigenvalues of u are 1).

Chevalley's commutator formula

Theorem

Let $G = \mathfrak{L}(K)$ be a Chevalley group over an arbitrary field K. Let r, s be linearly independent roots of $\mathfrak{L}$ and let t, u be elements of K.
Define the commutator

$$[x_s(u), x_r(t)] = (x_s(u))^{-1}(x_r(t))^{-1}x_s(u)x_r(t).$$

Then we have

$$[x_s(u), x_r(t)] = \prod_{i,j>0} x_{ir+js}(C_{ijrs}(-t)^i u^j),$$

where the product is taken over all pairs of positive integers i, j for which $ir + js$ is a root, in order of increasing $i + j$. Each of the constants C_{ijrs} is one of ± 1, ± 2, ± 3.

Theorem

Let $G = \mathfrak{L}(K)$ be a Chevalley group and let U be the subgroup generated by the root subgroups X_r with $r \in \Phi^+$. Then

1. *U is nilpotent*
2. *Each element of U is uniquely expressible in the form*

$$\prod_{r_i \in \Phi^+} x_{r_i}(t_i),$$

where the product is taken over all positive roots in increasing order.

Fix a root $r \in \Phi$. Recall that $SL_2(K)$ is the group of 2×2 matrices over the field K with determinant 1.

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Theorem

Let K be any field. Then there is a homomorphism ϕ_r from $SL_2(K)$ onto the subgroup $\langle X_r, X_{-r} \rangle$ of $\mathfrak{L}(K)$ under which

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_r(t),$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-r}(t).$$

Let $h_r(\lambda)$ denote the image of the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

under the homomorphism ϕ_r from $SL_2(K)$ onto subgroup $\langle X_r, X_{-r} \rangle$. We define H to be the subgroup of $\mathfrak{L}(K)$ generated by the elements $h_r(\lambda)$ for all $r \in \Phi$, $\lambda \neq 0 \in K$.

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$h_r(\lambda)$ operates on the Chevalley basis of $\mathfrak{L}_K$ by

$$h_r(\lambda).h_s = h_s, \qquad h_r(\lambda).e_s = \lambda^{A_{rs}} e_s.$$

So each element of H is an automorphism of $\mathfrak{L}_K$ which operates trivially on $\mathfrak{H}_K$ and transforms each root vector e_s into a multiple of itself.

We have the following facts about the subgroup H:

- H normalises each root subgroup X_r
- H normalises each of U and V
- Hence UH and VH are both subgroups of $\mathfrak{L}(K)$
- $UH \cap V = 1$
- $VH \cap U = 1$
- $UH \cap VH = H$.

Let n_r denote the image of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

under the homomorphism ϕ_r from $SL_2(K)$ onto subgroup $\langle X_r, X_{-r} \rangle$. We define N to be the subgroup of $\mathfrak{L}(K)$ generated by H and the elements n_r for all $r \in \Phi$.

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n_r operates on the Chevalley basis of $\mathfrak{L}_K$ by

$$n_r.h_s = h_{w_r(s)}, \qquad n_r.e_s = \pm e_{w_r(s)}.$$

So the element n_r of the Chevalley group $\mathfrak{L}(K)$ is closely related to the element w_r of the Weyl group W.

Theorem

There is a homomorphism from N onto W with kernel H under which $n_r \mapsto w_r$ for all $r \in \Phi$. Thus H is a normal subgroup of N and N/H is isomorphic to W.

- Let B denote the subgroup UH
- $\mathfrak{L}(K) = BNB$ (This is called the *Bruhat decomposition*)
- For each subset J of Π, let W_J be the subgroup of W generated by the w_i for $i \in J$ and let N_J be the subgroup mapping to W_J under the natural homomorphism. Then $P_J = BN_JB$ is a subgroup of $\mathfrak{L}(K)$
- There is a 1-1 correspondence between the double cosets of B in G and the elements of W
- We define a *parabolic subgroup* of $\mathfrak{L}(K)$ to be one that contains some conjugate of B.
- The subgroups P_J are the only subgroups of $\mathfrak{L}(K)$ containing B (so every parabolic subgroup of $\mathfrak{L}(K)$ is isomorphic to some P_J).
- Distinct subgroups P_J, P_K cannot be conjugate in $\mathfrak{L}(K)$.
- We have $P_J \cap P_K = P_{J \cap K}$. Thus the subgroups P_J form a lattice isomorphic to the lattice of subsets of Π.

Theorem

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ and K be an arbitrary field. Then the (adjoint) Chevalley group $\mathfrak{L}(K)$ is simple, except for the cases $A_1(2)$, $A_1(3)$, $B_2(2)$ and $G_2(2)$.
Every (adjoint) Chevalley group (even a non-simple one) has trivial centre.

Theorem

Let $\mathfrak{L}$ be a simple Lie algebra with $\mathfrak{L} \neq A_1$ and let K be a field. For each root r of $\mathfrak{L}$ and each element t of K introduce a symbol $\bar{x}_r(t)$. Let $\bar{G}$ be the abstract group generated by the elements $\bar{x}_r(t)$ subject to relations

$$\begin{aligned}
\bar{x}_r(t_1)\bar{x}_r(t_2) &= \bar{x}_r(t_1 + t_2), \\
[\bar{x}_s(u), \bar{x}_r(t)] &= \prod_{i,j>0} \bar{x}_{ir+js}(C_{ijrs}(-t)^i u^j), \\
\bar{h}_r(t_1)\bar{h}_r(t_2) &= \bar{h}_r(t_1 t_2), \qquad t_1 t_2 \neq 0, \\
\textit{where} \quad \bar{h}_r(t) &= \bar{n}(t)\bar{n}_r(-1) \\
\textit{and} \quad \bar{n}_r(t) &= \bar{x}_r(t)\bar{x}_{-r}(-t^{-1})\bar{x}_r(t).
\end{aligned}$$

Let $\bar{Z}$ be the centre of $\bar{G}$. Then $\bar{G}/\bar{Z}$ is isomorphic to the Chevalley group $G = \mathfrak{L}(K)$.