Lecture 1

Discrete Gaussian Free Field & scaling limits

In this lecture we will define the main object of interest in this course: the two-dimensional Discrete Gaussian Free Field (henceforth abbreviated as DGFF). We will then identify the properties that make this object special, and as a result also more amenable to analysis. This will particularly include a proof of the Gibbs-Markov property and the resulting connection to discrete harmonic analysis. We will then use the latter connection to infer the existence of a scaling limit to the continuum Gaussian Free Field (CGFF) which is a concept that will underlie, albeit often in disguise, most of the results discussed in the course.

1.1. Definitions

Let $d \geq 1$ be an integer and let $\mathbb{Z}^d$ denote the d-dimensional hypercubic lattice. This is a graph with vertices at all points of $\mathbb{R}^d$ with integer coordinates and an edge between any two vertices at unit (Euclidean) distance. Denoting the sets of edges (with both orientations identified as one edge) by $E(\mathbb{Z}^d)$, we then put forward:

Definition 1.1 [DGFF, explicit formula] *Let $V \subset \mathbb{Z}^d$ be finite. The DGFF in V is a process $\{h_x \colon x \in \mathbb{Z}^d\}$ indexed by vertices of $\mathbb{Z}^d$ with the law given (for any measurable $A \subseteq \mathbb{R}^{\mathbb{Z}^d}$) by*

$$P(h \in A) := \frac{1}{\text{norm.}} \int_A \mathrm{e}^{-\frac{1}{4d}\sum_{(x,y)\in \mathrm{E}(\mathbb{Z}^d)}(h_x - h_y)^2} \prod_{x\in V} \mathrm{d}h_x \prod_{x\notin V} \delta_0(\mathrm{d}h_x)\,. \tag{1.1}$$

Here δ_0 is the Dirac point-mass at 0*. The term "*norm.*" denotes a normalization constant.*

Note that this definition forces all values of h outside V to zero — we thus talk about *zero boundary condition*. To see that this definition is good, we pose:

Exercise 1.2 *Prove that the integral in* (1.1) *is finite for $A := \mathbb{R}^{\mathbb{Z}^d}$ and so the measure can be normalized to be a probability.*

The appearance of $4d$ factor in the exponent is a convention used in probability;

physicists would write $\frac{1}{2}$ instead of $\frac{1}{4d}$. Without this factor, the definition extends readily from $\mathbb{Z}^d$ to any locally-finite graph but since $\mathbb{Z}^d$ (in fact $\mathbb{Z}^2$) will be our main focus, we refrain from doing so.

The above definition gives the law of the DGFF in the form of a *Gibbs measure*, i.e., a measure of the form

$$\frac{1}{Z}\,\mathrm{e}^{-\beta H(h)}\,\nu(\mathrm{d}h) \tag{1.2}$$

where Z is a normalization constant, H is the Hamiltonian, β is the inverse temperature and ν is an *a priori* (typically a product) measure. Many models of statistical mechanics are cast this way; a key feature of the DGFF is that the Hamiltonian is a positive-definite quadratic form which makes the resulting law of h a multivariate Gaussian. This offers a possibility to define the law directly by prescribing the mean and covariances.

Let $\{X_n\colon n \ge 0\}$ denote the simple symmetric random walk on $\mathbb{Z}^d$. For $V \subset \mathbb{Z}^d$, we recall the definition of *Green function* in V given by

$$G^V(x,y) := E^x\Biggl(\sum_{n=0}^{\tau_{V^{\mathrm{c}}}-1} 1_{\{X_n=y\}}\Biggr) \tag{1.3}$$

where E^x is the expectation with respect to the law of X with $X_0 = x$ a.s. and $\tau_{V^{\mathrm{c}}} := \inf\{n \ge 0\colon X_n \notin V\}$ is the first exit time of the walk from V. We note:

Exercise 1.3 *Prove that, for any $x, y \in \mathbb{Z}^d$,*

$$V \mapsto G^V(x,y) \text{ is non-decreasing with respect to set inclusion.} \tag{1.4}$$

In particular, for any $V \subsetneq \mathbb{Z}^d$ (in any $d \ge 1$), we have $G^V(x,y) < \infty$ for all $x, y \in \mathbb{Z}^d$.

As is also easy to check, we also have $G^V(x,y) = 0$ unless $x, y \in V$ (in fact, unless x and y can be connected by a path on $\mathbb{Z}^d$ that lies entirely in V). The following additional properties of G^V will be important in the sequel:

Exercise 1.4 *Let Δ denote the discrete Laplacian on $\mathbb{Z}^d$ acting on $f\colon \mathbb{Z}^d \to \mathbb{R}$ as*

$$\Delta f(x) := \sum_{y\colon (x,y)\in \mathsf{E}(\mathbb{Z}^d)} \bigl(f(y) - f(x)\bigr). \tag{1.5}$$

Then for any $V \subsetneq \mathbb{Z}^d$ and any $x \in \mathbb{Z}^d$, $y \mapsto G^V(x,y)$ is the solution to

$$\begin{cases} \Delta G^V(x,\cdot) = -2d\delta_x(\cdot), & \text{on } V,\\ G^V(x,\cdot) = 0, & \text{on } V^{\mathrm{c}}, \end{cases} \tag{1.6}$$

where δ_x is the Kronecker delta at x.

Another way to phrase this exercise is by saying that the Green function is $(2d)$-multiple of the inverse of the negative Laplacian in $\ell^2(V)$ — with Dirichlet boundary conditions on V^{c}. This functional-analytic representation of the Green function allows us to readily solve:

Exercise 1.5 *For any $V \subsetneq \mathbb{Z}^d$ we have:*

(1) for any $x, y \in \mathbb{Z}^d$,

$$G^V(x,y) = G^V(y,x) \tag{1.7}$$

(2) for any $f\colon \mathbb{Z}^d \to \mathbb{R}$ with finite support,

$$\sum_{x,y\in\mathbb{Z}^d} G^V(x,y)f(x)f(y) \ge 0. \tag{1.8}$$

We remark that purely probabilistic solutions (i.e., those based solely on considerations of random walks) exist as well. What matters for us is that properties (1-2) make G^V a covariance of a Gaussian process. This gives rise to:

Definition 1.6 [DGFF, via the Green function] *Let $V \subsetneq \mathbb{Z}^d$ be given. The DGFF in V is a multivariate Gaussian process $\{h_x^V\colon x \in \mathbb{Z}^d\}$ with law determined by*

$$E(h_x^V) = 0 \quad \textit{and} \quad E(h_x^V h_y^V) = G^V(x,y), \qquad x, y \in \mathbb{Z}^d \tag{1.9}$$

or, written concisely,

$$h^V := \mathcal{N}(0, G^V) \tag{1.10}$$

In order to avoid accusations of logical inconsistency, we immediate pose:

Exercise 1.7 *Prove that for $V \subset \mathbb{Z}^d$ finite, Definitions 1.1 and 1.6 coincide.*

The advantage of Definition 1.6 over Definition 1.1 is that it works for infinite V as well. The functional-analytic connection can be pushed further as follows. Given a finite $V \subset \mathbb{Z}^d$, consider the Hilbert space $\mathcal{H}^V := \{f\colon \mathbb{Z}^d \to \mathbb{R},\ \mathrm{supp}(f) \subset V\}$ endowed with the *Dirichlet inner product*

$$\langle f, g\rangle_\nabla := \frac{1}{2d}\sum_{x\in\mathbb{Z}^d} \nabla f(x)\cdot\nabla g(x) \tag{1.11}$$

where $\nabla f(x)$, the gradient of f at x, is the vector in $\mathbb{R}^d$ whose i-th component is $f(x+e_i) - f(x)$, for e_i the i-th unit vector in $\mathbb{R}^d$. (Since the supports of f and g are finite, the sum is effectively finite.) We then have:

Lemma 1.8 [DGFF as Hilbert-space Gaussian] *For the setting as above with V finite, let $\{\varphi_n\colon n = 1, \dots, |V|\}$ be an orthonormal basis in $\mathcal{H}^V$ and let $Z_1, \dots, Z_{|V|}$ be i.i.d. standard normals. Then $\{\widetilde{h}_x\colon x \in \mathbb{Z}^d\}$, where*

$$\widetilde{h}_x := \sum_{n=1}^{|V|} \varphi_n(x) Z_n, \qquad x \in \mathbb{Z}^d, \tag{1.12}$$

has the law of the DGFF in V.

Exercise 1.9 *Prove the previous lemma.*

Lemma 1.8 thus gives yet another way to define DGFF. (The restriction to finite V was imposed only for simplicity.) As we will see at the end of this lecture, this is the definition that generalizes seamlessly to a continuum underlying space.

We finish this section by a short remark on notation: Throughout these lectures we will write h^V to denote the whole configuration of the DGFF in V and write h^V_x for the value of h^V at x. We will occasionally write $h^V(x)$ instead of h^V_x.

1.2. Why $d = 2$ only?

Throughout this course we will focus on the case of $d = 2$. Let us therefore comment on what makes the two-dimensional DGFF special. We begin by noting:

Lemma 1.10 [Green function asymptotic] *Let $V_N := (0,N)^d \cap \mathbb{Z}^d$ and, for any $\epsilon \in (0,1/2)$, denote $V_N^\epsilon := (\epsilon N, (1-\epsilon)N)^d \cap \mathbb{Z}^d$. Then for any $x \in V_N^\epsilon$,*

$$G^{V_N}(x,x) \underset{N\to\infty}{\sim} \begin{cases} N, & d=1, \\ \log N, & d=2, \\ 1, & d\ge 3, \end{cases} \tag{1.13}$$

where "$\sim$" means that the ratio of the left and right-hand side tends to a positive and finite number as $N \to \infty$ (which may depend on where x is asymptotically located in V_N).

Proof (sketch). We will only prove this in $d = 1$ and $d \ge 3$ as the $d = 2$ case will be treated later in far more explicit terms. To this end, a routine application of the Strong Markov Property for the simple symmetric random walk yields

$$G^V(x,x) = \frac{1}{P^x(\hat\tau_x > \tau_{V^c})}, \tag{1.14}$$

where $\hat\tau_x := \inf\{n \ge 1 \colon X_n = x\}$ is the first *return* time to x. Assuming $x \in V$, in $d = 1$ we then have

$$P^x(\hat\tau_x > \tau_{V^c}) = \frac{1}{2}\Big[P^{x+1}(\tau_x > \tau_{V^c}) + P^{x-1}(\tau_x > \tau_{V^c})\Big] \tag{1.15}$$

where $\tau_x := \inf\{n \ge 0 \colon X_n = x\}$ is the first *hitting* time at x. The Markov property of the simple random walk shows that $y \mapsto P^y(\tau_x > \tau_{V_N^c})$ is discrete harmonic on $\{1,\dots,x-1\} \cup \{x+1,\dots,N-1\}$ with value zero at $y = x$ and one at $y = 0$ and $y = N$. Hence

$$P^{x+1}(\tau_x > \tau_{V^c}) = \frac{1}{N-x} \quad \text{and} \quad P^{x-1}(\tau_x > \tau_{V^c}) = \frac{1}{x}. \tag{1.16}$$

As $x \in V_N^\epsilon$, both of these probabilities are order $1/N$ and scale nicely when x grows proportionally to N. This proves the claim in $d = 1$.

In $d \ge 3$ we note that the transience and translation invariance of the simple symmetric random walk implies

$$G^{V_N}(x,x) \underset{N\to\infty}{\longrightarrow} \frac{1}{P^0(\hat\tau_0 = \infty)} \tag{1.17}$$

uniformly in the above x. (Transience is equivalent to $P^0(\hat{\tau}_0 = \infty) > 0$.) $\square$

Focusing for the moment on $d = 1$, the fact that the variance blows up linearly suggests that we normalize the DGFF by the square-root of the variance, i.e., $\sqrt{N}$, and attempt to extract a limit. This does work and yields:

Theorem 1.11 [Scaling to Brownian Bridge in $d = 1$] *Suppose $d = 1$ and let h be the DGFF in $V_N := (0, N) \cap \mathbb{Z}$. Then*

$$\Big\{ \frac{1}{\sqrt{N}} h_{\lfloor tN \rfloor} \colon t \in [0,1] \Big\} \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \big\{ \sqrt{2}\, W_t \colon t \in [0,1] \big\} \tag{1.18}$$

where W is the standard Brownian Bridge on $[0,1]$.

We leave it to the reader to solve:

Exercise 1.12 *Prove Theorem 1.11. [Hint: Note that $\{h_{x+1} - h_x \colon x = 0, \dots, N-1\}$ are i.i.d. $\mathcal{N}(0,2)$ conditioned on their total sum being zero.]*

We remark that the limit taken in Theorem 1.11 is an example of a *scaling (or continuum) limit* — the lattice spacing is taken to zero while keeping the overall (continuum) domain fixed. In renormalization group theory, taking the scaling limit corresponds to the removal of an ultraviolet cutoff.

Moving to $d \geq 3$, in the proof of Lemma 1.10 we observed enough to conclude:

Theorem 1.13 [Full-space limit in $d \geq 3$] *Suppose $d \geq 3$ and let h be the DGFF in $\widetilde{V}_N := (-N/2, N/2)^d \cap \mathbb{Z}^d$. Then for any $x, y \in \mathbb{Z}^d$,*

$$G^{\widetilde{V}_N}(x,y) \underset{N\to\infty}{\longrightarrow} G^{\mathbb{Z}^d}(x,y) := \int \frac{\mathrm{d}k}{(2\pi)^d} \frac{\cos(k \cdot (x-y))}{\frac{2}{d}\sum_{j=1}^d \sin(k_j/2)^2} \tag{1.19}$$

In particular, $h^{\widetilde{V}_N} \to \mathcal{N}(0, G^{\mathbb{Z}^d})$:= full space DGFF.

This means that the DGFF in large-enough (finite) domains is well approximated by the full-space DGFF as long as we are far away from the boundary. This is an example of a *thermodynamic limit* — the lattice stays fixed and the boundaries of domains slide off to infinity. In renormalization group theory, taking the thermodynamic limit corresponds to the removal of an infrared cutoff.

From Lemma 1.10 it is clear that the thermodynamic limit is meaningless for the two-dimensional DGFF (indeed, variances blow up and, since we are talking about Gaussian random variables, there is no tightness). Let us attempt to take the scaling limit just as in $d = 1$: normalize the field by the square-root of the variance (i.e., $\sqrt{\log N}$) and extract a distributional limit (for which it suffices to prove the limit of the covariances). Here we note that

$$\sup_{N\geq 1} \sup_{\substack{x,y\in V_N \\ |x-y|\geq \epsilon N}} G^{V_N}(x,y) < \infty, \tag{1.20}$$

a fact that we will prove later as well. For any $s,t \in (0,1)^2$ we thus get

$$E\biggl(\frac{h^{V_N}_{\lfloor tN\rfloor}}{\sqrt{\log N}},\frac{h^{V_N}_{\lfloor sN\rfloor}}{\sqrt{\log N}}\biggr)\ \underset{N\to\infty}{\longrightarrow}\ \begin{cases} c(t)>0, & \text{if } s=t,\\ 0, & \text{else,}\end{cases} \tag{1.21}$$

where, here and henceforth,

$$\lfloor tN\rfloor := \text{the unique } z\in\mathbb{Z}^2 \text{ such that } \ tN\in z+[0,1)^2\,. \tag{1.22}$$

The only way to read this is that the limit distribution is a collection of independent normals indexed by $t\in(0,1)^2$ — an object too irregular and generic to retain useful information from before the limit was taken.

As we will see, the right thing to do is to take the limit of the DGFF without any normalization. That will lead to a singular limit object as well but one that captures better the behavior of the DGFF and features additional symmetries (namely, conformal invariance) not present at the discrete level.

1.3. Green function asymptotic

Let us now analyze the situation in $d=2$ in more detail. Our aim is to consider the DGFF in *sequences* of lattice domains $\{D_N\colon N\ge 1\}$ that somehow correspond to the scaled-up version of a given continuum domain $D\subset\mathbb{C}$. The assumptions on the continuum domains are the subject of:

Definition 1.14 [Admissible domains] *Let $\mathfrak{D}$ be the class of sets $D\subset\mathbb{C}$ that are bounded, open and connected and such that their topological boundary ∂D is the union of a finite number of connected components each of which has a positive (Euclidean) diameter.*

For the sake of future use we note:

Exercise 1.15 *All bounded, open and simply connected $D\subset\mathbb{C}$ belong to $\mathfrak{D}$.*

As to what sequence of discrete approximations of D we will permit, a natural choice would be to work with plain discretization $\{x\in\mathbb{Z}^2\colon x/N\in D\}$. However, this is too crude in the sense that parts of ∂D could be missed out completely; see Fig. 1.1. We thus have to qualify admissible discretizations more precisely:

Definition 1.16 [Admissible lattice approximations] *A family $\{D_N\colon N\ge 1\}$ of subsets of $\mathbb{Z}^2$ is a* sequence of admissible lattice approximations *of D if*

$$D_N\subseteq \bigl\{x\in\mathbb{Z}^2\colon \mathrm{d}_\infty(x/N,D^{\mathrm{c}})>{}^1\!/\!_N\bigr\}\,, \tag{1.23}$$

where d_∞ denotes the ℓ^∞-distance on $\mathbb{Z}^2$, and if, for each $\delta>0$,

$$D_N\supseteq \bigl\{x\in\mathbb{Z}^2\colon \mathrm{d}_\infty(x/N,D^{\mathrm{c}})>\delta\bigr\} \tag{1.24}$$

holds once N is sufficiently large (depending possibly on δ).

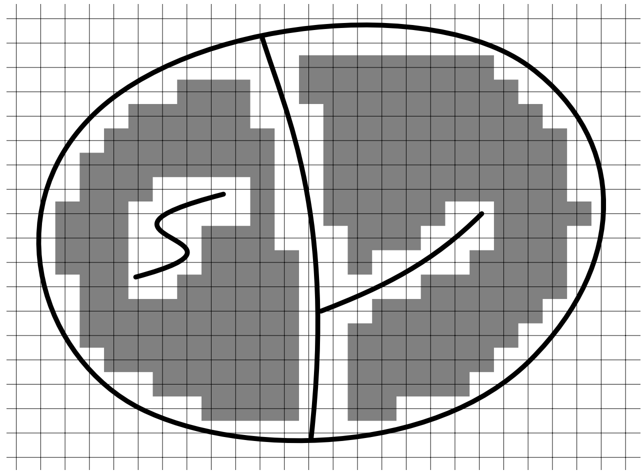

Figure 1.1: An example of discretization of a continuum domain. The continuum domain corresponds to the region in the plane bounded by the solid curves. The discrete domain is the set of vertices in the shaded areas.

Note that this still ensures that $x \in D_N$ implies $x/N \in D$. For the statement of the next theorem, consider the (standard) Brownian motion $\{B_t \colon t \geq 0\}$ on $\mathbb{R}^2$ and let $\tau_{D^c} := \inf\{t \geq 0 \colon B_t \notin D\}$ be the first exit time from D. Denote by $\Pi^D(x, \cdot)$, with

$$\Pi^D(x, A) := P^x\big(B_{\tau_{D^c}} \in A\big), \tag{1.25}$$

the law of the exit point from D of the Brownian motion started from x. This is also known as the *harmonic measure from x in D*. The following is then perhaps the most technical result in this lecture:

Theorem 1.17 [Green function asymptotic in $d = 2$] *Suppose $d = 2$, let $D \in \mathfrak{D}$ and let $\{D_N \colon N \geq 1\}$ be a sequence of admissible lattice approximations of D. Set $g := 2/\pi$. Then there is $c_0 \in \mathbb{R}$ such that for all $x \in D$,*

$$G^{D_N}\big(\lfloor xN \rfloor, \lfloor xN \rfloor\big) = g \log N + c_0 + g \int_{\partial D} \Pi^D(x, \mathrm{d}z) \log|x - z| + o(1) \tag{1.26}$$

and, for all $x, y \in D$ with $x \neq y$, also

$$G^{D_N}\big(\lfloor xN \rfloor, \lfloor yN \rfloor\big) = -g \log|x - y| + g \int_{\partial D} \Pi^D(x, \mathrm{d}z) \log|y - z| + o(1). \tag{1.27}$$

Here $o(1) \to 0$ as $N \to \infty$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

Proof (modulo two lemmas). The proof of the theorem starts by a convenient representation of the Green function using the *potential kernel*. This is a function $\mathfrak{a} \colon \mathbb{Z}^2 \to [0, \infty)$ defined, e.g., by the explicit formula

$$\mathfrak{a}(x) := \int_{(-\pi,\pi)^2} \frac{\mathrm{d}k}{(2\pi)^2} \, \frac{1 - \cos(k \cdot x)}{\sin(k_1/2)^2 + \sin(k_2/2)^2} \,. \tag{1.28}$$

A key property for us is the content of:

Exercise 1.18 *Show that* $\mathfrak{a}$ *solves the Poisson problem*

$$\begin{cases} \Delta\mathfrak{a}(\cdot) = 4\delta_0(\cdot), & \text{on } \mathbb{Z}^2, \\ \mathfrak{a}(0) = 0, \end{cases} \tag{1.29}$$

where, as before, δ_0 *is the Kronecker delta at* 0.

Writing ∂V for external vertex boundary of V, i.e., the set of vertices in V^{c} that have an edge to a vertex in V, we now get:

Lemma 1.19 [Green function from potential kernel] *For any finite* $V \subset \mathbb{Z}^2$ *and any* $x, y \in V$,

$$G^V(x,y) = -\mathfrak{a}(x-y) + \sum_{z\in\partial V} H^V(x,z)\mathfrak{a}(z-y)\,, \tag{1.30}$$

where $H^V(x,z) := P^x(X_{\tau_{V^c}} = z)$ *is the probability that the simple symmetric random walk started from* x *exists* V *at* $z \in \partial V$.

Proof. In light of (1.6), (1.29) and the translation invariance of the lattice Laplacian,

$$\phi(y) := G^V(x,y) + \mathfrak{a}(x-y) \tag{1.31}$$

is discrete harmonic in V. Hence $M_n := \phi(X_{\tau_{V^c}\wedge n})$ is a martingale for the usual filtration $\sigma(X_0, \dots, X_n)$. The finiteness of V ensures that M is bounded; the Optional Stopping Theorem then gives

$$\phi(y) = \sum_{z\in\partial V} H^V(y,z)\phi(z). \tag{1.32}$$

Since $G^V(x,\cdot)$ vanishes on ∂V, this and the symmetry of $x,y \mapsto G^V(x,y)$ and of $x,y \mapsto \mathfrak{a}(x-y)$ now readily imply the claim. □

We note that the restriction to finite V was not a mere convenience. Indeed:

Exercise 1.20 *Show that* $G^{\mathbb{Z}^2\smallsetminus\{0\}}(x,y) = \mathfrak{a}(x) + \mathfrak{a}(y) - \mathfrak{a}(x-y)$. *Use this to conclude that, in particular,* (1.30) *is generally false for infinite* V.

As the next step of the proof, we invoke:

Lemma 1.21 [Potential kernel asymptotic] *There is* $c_0 \in \mathbb{R}$ *such that*

$$\mathfrak{a}(x) = g\log|x| + c_0 + O\big(|x|^{-2}\big), \tag{1.33}$$

where, we recall, $g = 2/\pi$ *and* $|x|$ *is the Euclidean norm of* x.

This asymptotic form was apparently first proved by Stöhr in 1950. In a 2004 paper, Kozma and Schreiber analyzed the behavior of the potential kernel on other lattices and identified the constants g and c_0 in terms of specific geometric attributes of the lattice. In our case, we can certainly attempt:

Exercise 1.22 *Prove Lemma 1.21 by suitable asymptotic analysis of* (1.28).

Using (1.30) and (1.33) together, for $x, y \in D$ with $x \neq y$ we now get

$$\begin{aligned} G^{D_N}\big(\lfloor xN\rfloor, \lfloor yN\rfloor\big) = g\log|x-y| \\ + g\sum_{z\in\partial D_N} H^{D_N}\big(\lfloor xN\rfloor, z\big)\log|y-z/N| + O(1/N) \end{aligned} \tag{1.34}$$

where the $O(1/N)$ term arises from the approximation of (the various occurrences of) $\lfloor xN\rfloor$ by xN and also from the error in (1.33). To get (1.27), we thus need to convert the sum into the integral. Here we will need:

Lemma 1.23 [Weak convergence of harmonic measures] *For any $D \in \mathfrak{D}$ and any sequence $\{D_N : N \geq 1\}$ of admissible lattice approximations of D,*

$$\sum_{x\in\partial D_N} H^{D_N}\big(\lfloor xN\rfloor, z\big)\,\delta_{z/N}(\cdot) \;\underset{N\to\infty}{\overset{\text{weakly}}{\longrightarrow}}\; \Pi^D(x,\cdot). \tag{1.35}$$

We will omit the proof as it would take us too far on a tangent; the reader is instead referred to (the Appendix of) a joint paper with O. Louidor (arXiv:1410.4676). The idea is to use Donsker's Invariance Principle to extract a coupling of the simple random walk and the Brownian motion that ensures that once the random walk exits D_N, the Brownian motion will exit D at a "nearby" point. This is where we find it useful that the boundary components to have a non-trivial diameter.

Since $u \mapsto \log|y-u|$ is bounded and continuous in a neighborhood of ∂D (whenever $y \in D$), the weak convergence in Lemma 1.23 implies

$$\sum_{z\in\partial D_N} H^{D_N}\big(\lfloor xN\rfloor, z\big)\log|y-z/N| = \int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|y-z| + o(1)\,, \tag{1.36}$$

with $o(1) \to 0$ as $N \to \infty$. This proves (1.27). The proof of (1.26) only amounts to the substitution of $-g\log|x-y|$ in (1.34) by $g\log N + c_0$; the convergence of the sum to the integral is then handled as before. □

We remark that the convergence of discrete harmonic measure to the continuous one in Lemma 1.23 is where all of the above assumptions on the continuum domain and its lattice approximations enter.

The objects appearing on the right-hand side of (1.26–1.27) are actually quite well known. Indeed, we set:

Definition 1.24 [Continuum Green function & conformal radius] *For $D \subset \mathbb{C}$ bounded and open, we define the continuum Green function in D from x to y by*

$$\widehat{G}^D(x,y) := -g\log|x-y| + g\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|y-z| \tag{1.37}$$

Similarly, for $x \in D$ we define

$$r_D(x) := \exp\Big\{\int_{\partial D}\Pi^D(x,\mathrm{d}z)\log|x-z|\Big\} \tag{1.38}$$

to be the conformal radius of D from x.

The continuum Green function is usually defined as the fundamental solution to the Poisson equation; i.e., the continuum version of (1.6). We will not need that fact in what follows so we will content ourself with the explicit form above. Similarly, for open simply connected $D \subset \mathbb{C}$, the conformal radius of D from x is defined as the value $|f'(x)|^{-1}$ for f any conformal bijection of D onto $\{z \in \mathbb{C}: |z| < 1\}$ such that $f(x) = 0$. (The result does not depend on the choice of the bijection.) Although we will not need this connection either, it is instructive to solve:

Exercise 1.25 *Check that this coincides with $r_D(x)$ above.*

For this as well as later derivations it may be useful to know that the harmonic measure is *conformally invariant*: If $f: D \mapsto f(D)$ is a conformal bijection with continuous extensions of f to ∂D and f^{-1} to $\partial f(D)$, then

$$\Pi^D(x, A) = \Pi^{f(D)}\big(f(x), f(A)\big) \tag{1.39}$$

for any measurable $A \subset \partial D$. This can be seen as a direct consequence of conformal invariance of the Brownian motion, although more straightforward approaches to prove this exist as well.

1.4. Continuum Gaussian Free Field

Theorem 1.17 reveals two important facts: First, as already observed in Lemma 1.10 without proof, the pointwise limit of unscaled DGFF is meaningless as, in light of (1.26), there is no tightness. Second, by (1.27), the off-diagonal covariances of the DGFF do have a limit which is given by the continuum Green function. This function is singular on the diagonal, but the singularity is only logarithmic and thus pretty mild. This permits us to derive:

Theorem 1.26 *Suppose $D \in \mathfrak{D}$ and let $\{D_N: N \geq 1\}$ be any admissible sequence of lattice approximations of D. For any bounded measurable $f: D \to \mathbb{R}$, let*

$$h^{D_N}(f) := \int_D \mathrm{d}x\, f(x) h^{D_N}_{\lfloor xN \rfloor}. \tag{1.40}$$

Then

$$h^{D_N}(f) \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{N}(0, \sigma_f^2)\,, \tag{1.41}$$

where

$$\sigma_f^2 := \int_{D\times D} \mathrm{d}x\mathrm{d}y\, f(x)f(y)\widehat{G}^D(x,y)\,. \tag{1.42}$$

Proof. The random variable $h^{D_N}(f)$ is Gaussian with mean zero and variance

$$E\big[h^{D_N}(f)^2\big] = \int_{D\times D} \mathrm{d}x\mathrm{d}y\, f(x)f(y)G^{D_N}\big(\lfloor xN \rfloor, \lfloor yN \rfloor\big). \tag{1.43}$$

The reasoning underlying the proof of Theorem 1.17 yields the existence of $\tilde{c} \in \mathbb{R}$ such that for all N large enough and all $x, y \in D_N$,

$$G^{V_N}(x,y) \leq g \log\left(\frac{N}{|x-y| \vee 1}\right) + \tilde{c}. \tag{1.44}$$

This shows

$$G^{D_N}\big(\lfloor xN\rfloor,\lfloor yN\rfloor\big)\le c\log|x-y| \tag{1.45}$$

for some constant $c\in(0,\infty)$ uniformly in $x,y\in D$. Using this bound, we can estimate (and later neglect) the contributions to the integral in (1.43) from pairs (x,y) satisfying $|x-y|<\epsilon$ for any given $\epsilon>0$. The convergence of the remaining part of the integral is then treated using the pointwise convergence (1.27) and the Dominated Convergence Theorem. □

Exercise 1.27 *Give a full proof of the bound* (1.44)*.*

The definition (1.40) is tantamount to a projection of the field configuration onto a test function. Theorem 1.26 then states that these projections admit a continuum limit. This suggests we could treat the limit random variable as a *linear functional* on a suitable space of test functions. This underlies:

Definition 1.28 [CGFF as a function-space indexed Gaussian] *A continuum Gaussian Free Field (CGFF) on a bounded, open $D\subset\mathbb{C}$ is an assignment $f\mapsto\Phi(f)$ of a random variable to each bounded measurable $f\colon D\to\mathbb{R}$ such that*

(1) Φ is a.s. linear, i.e.,

$$\Phi(af+bg)=a\Phi(f)+b\Phi(g)\quad\text{a.s.} \tag{1.46}$$

for each bounded measurable f and g and each $a,b\in\mathbb{R}$, and

(2) for each bounded measurable f,

$$\Phi(f)\overset{\text{law}}{=}\mathcal{N}(0,\sigma_f^2), \tag{1.47}$$

where σ_f^2 is as in (1.42)*.*

Theorem 1.26 shows that the DGFF scales to the CGFF taken in this sense — which also means that a CGFF as defined above exists! (The linearity is immediate from (1.40).) We remark that definitions given in the literature usually require that the CGFF is even a.s. *continuous* in suitable topology on a suitable (and suitably completed) class of functions. Such continuity considerations are important when one tries to assign meaning to $\Phi(f)$ for singular f (e.g., those vanishing Lebesgue a.e.) or for f varying continuously with respect to some parameter. This is for example useful in the study of the *disc-average process* $t\mapsto\Phi(f_t)$ where

$$f_t(y):=\frac{1}{\pi\mathrm{e}^{-2t}}1_{B(x,\mathrm{e}^{-t})}(y)\quad\text{for}\quad B(x,r):=\big\{y\in\mathbb{C}\colon|y-x|<r\big\}. \tag{1.48}$$

Here it is quite instructive to note:

Exercise 1.29 *For CGFF as defined above and for any $x\in D$ at Euclidean distance $r>0$ from D^{c}, show that for $t>\log r$, the process $t\mapsto\Phi(f_t)$ has independent increments with*

$$\mathrm{Var}\big(\Phi(f_t)\big)=gt+c_1+g\log r_D(x) \tag{1.49}$$

for some $c_1\in\mathbb{R}$ independent of x. In particular, $t\mapsto\Phi(f_t)$ admits a continuous version whose law is that of a Brownian motion.