Topics on the two-dimensional Gaussian Free Field

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Contents

The goal of this series of lectures is to describe some aspects of the Gaussian Free Field (GFF in the sequel) in the continuum and to provide a gentle self-contained introduction and presentation of some recent developments on this topic, notably by Schramm, Sheffield and Miller.

As we shall see, the GFF is a very natural generalization of Brownian Motion when one replace one-dimensional time by a multidimensional parameter. In the study of Brownian motion and of stochastic calculus, a very important idea was that of stopping times and of the strong Markov property of the Brownian motion. This relies of course very much on the fact that the time-line is naturally ordered (past vs. future). Another key-feature that enables to define interesting stopping times such as "the first time at which the Brownian path exits a given open set" is that Brownian paths are almost surely continuous (or more precisely, they almost surely have a continuous version).

When one replaces time-intervals by a multidimensional open set, two important problems show up when one tries to generalize this strong Markov property:

- The Gaussian Free Field is not a continuous function anymore. As we shall see, it is only a "generalized function", so that trying to generalize "first exit times" seems difficult.
- There is no natural "time-orientation" so that it is not clear how to generalize the notion of stopping times.

We shall try to describe ways to address these two questions. This will give rise to the notion of "local sets" of the Gaussian Free Field as introduced by Schramm and Sheffield, and provides (in the two-dimensional case) a natural approach to some of the Stochastic Loewner Evolutions.

Along the way, we will try to present in a simple way various ideas that are related to the GFF in the discrete and in the continuous setting: We will briefly discuss occupation times of loop-soups, determinants of the Laplacian, uniform spanning trees etc.

MISSING ITEMS IN THIS PRELIMINARY VERSION:

List of references, discussions of bibliography etc.

In Chapter 3: CLE4 construction and coupling with the GFF, properties of this coupling, CLE4 and the square of the GFF, CLE4 and Liouville QG. Bibliographic references to Schramm-Sheffield, Sheffield, Miller-Sheffield, Dubédat etc.

Probably lots of typos and some mistakes here and there...

Chapter 1

A few things about the discrete GFF

In this first chapter that corresponds the first four lectures of the course, we recall a few rather basic fact about the discrete Gaussian Free Field in order to guide our intuition about what will happen in the continuous case. We will also give some details about the relation between random collection of loops and the GFF.

Our goal here will not be to strive to generality, but to focus on the essentials (we will in particular not study the case of general Markov chains, but only on simple random walks).

1.1 Some basic features

1.1.1 Green's function basics

Take $d \geqslant 1$ and D a finite subset $\mathbb{Z}^d$ with N sites. Define

$$\partial D := \{x \in \mathbb{Z} : d(x, D) = 1\} \text{ and } \bar{D} := D \cup \partial D.$$

We denote by $\mathcal{F}_{(D)}$ the set of functions from $\mathbb{Z}^d$ into $\mathbb{R}$ that are equal to 0 outside of D.

1. Discrete Laplacian: When $F : D \mapsto \mathbb{R}^d$, we define for all $x \in D$,

$$\Delta F(x) := \frac{1}{2d} \sum_{y \,:\, y \sim x} (F(y) - F(x)),$$

where here and in the sequel, the sum $\sum_{y \,:\, y \sim x}$ means that we sum over the $2d$ neighbors of x in $\mathbb{Z}^d$. Mind that we do not care about the value of Δ outside of D (in particular on ∂D).

Clearly, we can view Δ as a linear operator from $\mathcal{F}_{(D)}$ into $\mathbb{R}^D$. It is easy to check that Δ is injective using the maximum principle: If $\Delta F = 0$, then choose $\bar{x} \in D$ so that $F(\bar{x}) = \max_{x \in D} F(x)$, and because $\Delta F(\bar{x}) = 0$, this implies that the value of F on all the neighbors of $\bar{x}$ are all equal to $F(\bar{x})$. But then, this holds also for all neighbors of neighbors of $\bar{x}$. But, since D is finite, this means that we will eventually find a boundary point y for which $F(y) = F(\bar{x})$ from which it follows that $F(\bar{x}) = 0$. Applying the same reasoning to $-F$, we readily conclude the $F = 0$ in D.

Hence, Δ is a bijective linear map from the $\#D$-dimensional vector space $\mathcal{F}_D$ onto $\mathbb{R}^D$. This means that there exists a linear inverse map. For any choice of function $u : D \to \mathbb{R}$, there exists exactly one function $F \in \mathcal{F}_D$ such that $\Delta F(x) = u(x)$ for all $x \in D$.

2. Random walks and Green's function: Let $(X_n)_{n \geqslant 0}$ be a simple random walk in $\mathbb{Z}^d$ started at x under the probability measure P_x. Let $\tau = \tau_D := \inf\{n \geqslant 0 : X_n \notin D\}$ be its first exit time of D. We can define the Green function in D on $D \times D$ as

$$G_D(x, y) := E_x \left(\sum_{n=0}^{\tau-1} \mathbf{1}_{\{X_n = y\}} \right).$$

Let us make a few simple remarks:

- By convention, we will set $G_D(x, y) = 0$ as soon as one of the two points x or y is not in D.
- G_D is symmetric in x and y. Indeed,

$$\begin{aligned} G_D(x, y) &= E_x \left(\sum_{n \geqslant 0} \mathbf{1}_{\{x_n = y, n < \tau\}} \right) \\ &= \sum_{n \geqslant 0} P_x(X_n = y, n < \tau) \\ &= \sum_{n \geqslant 0} \#\{\text{paths } x \to y \text{ in } n \text{ steps within } D\} \times \left(\frac{1}{2d} \right)^n \end{aligned}$$

  and this last expression is clearly symmetric in x and y (the number of paths from x to y with n steps in D is equal to the number of paths from y to x).

- For all $x \in D$, let us define $g_{D,x}(\cdot) = G_D(x, \cdot)$. This is a function in $\mathcal{F}_{(D)}$ (it is equal to zero outside of D). We see that for all $y \neq x$ in D, $\Delta g_{D,x}(y) = 0$ because:

$$G_D(x, y) = G_D(y, x) = E_y \left(\sum_{n \geqslant 1} \mathbf{1}_{\{X_n = x, n < \tau\}} \right) = \sum_{z \sim y} \frac{1}{2d} G_D(z, x)$$

  where we have used the Markov property at time 1. Also, the very same observation (but noting that at time 0, the random walk starting at x is at x) shows that $\Delta g_{D,x}(x) = -1$. Hence, $g_{D,x}$ is a function in $\mathcal{F}_D$ so that

$$\Delta g_{D,x}(y) = -\mathbf{1}_{\{x = y\}}$$

  for all y in D. As we have noted above, it is in fact the unique function in $\mathcal{F}_D$ with this property.

  Let us now consider any function $F \in \mathcal{F}_{(D)}$. Clearly,

$$F(\cdot) = -\sum_{x \in D} \Delta F(x) g_{D,x}(\cdot),$$

  because the Laplacian of both these functions (that belong to $\mathcal{F}_D$) coincide on D. This shows in particular that the matrix $-G_D$ can be viewed as the inverse of the Laplacian.

3. There is a nice scalar product that one can define on $\mathcal{F}_{(D)}$

$$(F_1, F_2) = \frac{1}{2} \times \frac{1}{2d} \times \sum_{x \sim y} (F_1(y) - F_1(x))(F_2(y) - F_2(x)),$$

$$\|F\|^2 = \frac{1}{2d} \times \frac{1}{2} \times \sum_{x \sim y} (F(y) - F(x))^2.$$

Mind that the sum $\sum_{x \sim y}$ here is over all pairs (x, y) in $\mathbb{Z}^d$ such that $x \sim y$ (i.e. x and y are neighbors). In other words, each "edge" is counted twice (hence the $1/2$ term to compensate for this!). We have that

$$(F_1, F_2) = \frac{1}{2d} \times \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} -F_1(x)(F_2(y) - F_2(x)) = -\sum_{x \in D} F_1(x) \Delta F_2(x).$$

By symmetry, this expression is therefore also equal to $-\sum_{x \in D} F_2(x) \Delta F_1(x)$.

We would like to stress that in the sequel, we will always use the notation (F_1, F_2) to denote this scalar product between functions supported on a finite set. In some texts, this scalar product is sometimes referred to as $(F_1, F_2)_\nabla$ in order to make the difference with the usual $\sum_x F_1(x)F_2(x)$, but since we will basically not use the latter, we will drop this subscript ∇.

If we consider a fixed $K \subseteq D$, we can note that $\mathcal{F}_K \subset \mathcal{F}_D$. We can also define the set of functions $\mathrm{Harm}(K, D)$ to be the set of functions in $\mathcal{F}_D$ that are harmonic in K i.e. such that for all $x \in K$, $\Delta F(x) = 0$. Note that this is a vector space with dimension $\#D - \#K$ (indeed, a function in that space will be characterized uniquely once one chooses $\Delta F(x)$ for all $x \in D \setminus K$, and one can choose these values as one wishes). Furthermore, the previous expression shows that if $F_1 \in \mathcal{F}_K \subset \mathcal{F}_D$ and $F_2 \in \mathrm{Harm}(K, D)$, then $(F_1, F_2)_D = 0$ because for each $x \in D$, either $F_1(x) = 0$ of $\Delta F_2(x) = 0$. Hence, the two spaces $\mathcal{F}_K$ and $\mathrm{Harm}(K, D)$ are orthogonal. This shows in particular that for all $F \in \mathcal{F}_D$, there exists a unique pair of functions $F_K \in \mathcal{F}_K$ and $F^K \in \mathrm{Harm}(K, D)$ so that $F = F_K + F^K$.

There a final simple remark that one can make, and that will be useful in the next definition. Suppose that u is any function in $\mathcal{F}_D$, and define as in 1., the function $F \in \mathcal{F}_D$ such that $\Delta F = u$ in D. Then,

$$\sum_{x,y} u(x)u(y)G_D(x, y) = -\sum_x \Delta F(x)F(x) = (F, F) \geq 0.$$

1.1.2 Discrete Gaussian Free Field basics

We say that the family of random variables $(\Gamma(x))_{x \in D}$ is a **Gaussian Free Field in** D, if it is a centered Gaussian vector with covariance given by $E(\Gamma(x)\Gamma(y)) = G_D(x, y)$.

Mind that this is the GFF in D viewed as a subset of $\mathbb{Z}^d$ (so the position of the points in ∂D play a role), and that it is also sometimes referred to as the GFF in D with zero boundary conditions.

The theory of Gaussian vectors shows us immediately that the distribution of a centered Gaussian vector is characterized by its covariance matrix. This works because we know (see the last displayed equation of the previous subsection) that $G_D(\cdot, \cdot)$ is indeed an admissible covariance matrix.

There are several simple ways to construct the discrete GFF:

1. One direct way is to define directly the density of $(\Gamma(x_1), ..., \Gamma(x_n))$ at $\gamma = (\gamma_1, ..., \gamma_n)$, using the fact that G_D is invertible (we have seen that the inverse matrix is $-\Delta$). The density is indeed equal to a constant $C(D)$ times

$$\exp\left\{-\frac{1}{2} \times \gamma \times G_D^{-1} \times \gamma^t\right\}$$

(here, $\times$ means the product of matrices, and γ^t the transposed column matrix) and to note (using the simple results that we recalled above) that this is equal to

$$\exp\left\{-\frac{1}{4(2d)} \sum_{x_i \sim x_j} (\gamma_i - \gamma_j)^2\right\}.$$

The change of variable formula (just use an orthonomal basis associated to the positive definite bilinear form $\sum_{x_i \sim x_j} (\gamma_i - \gamma_j)^2$ – mind that the boundary conditions ensure that it is definite) tells us also what the constant is in front of this expression, namely

$$C(D) = \frac{1}{(\sqrt{2\pi})^n \times \sqrt{\det(G_D)}},$$

where here, $\det(G_D)$ means the determinant of G_D viewed as a matrix. As $G_D^{-1} = -\Delta$ (when the latter is viewed as a linear map from $\mathcal{F}_D$ onto itself), we also know that $\det(-\Delta_{\mathcal{F}_D}) = 1/\det(G_D)$. This constant can be reinterpreted in various ways (and we will do this later on).

2. With the previous construction we can see that in one dimension, the GFF can be viewed as a random walk with Gaussian increments, conditioned to come back to 0 at the ends of the intervals in which it is defined. For instance, if $D = \{1, \dots, n-1\}$, then the density of $(\Gamma(1), \dots, \Gamma(n-1))$ at $(\gamma_1, \dots, \gamma_{n-1})$ is a multiple of

$$\exp\left(-(1/2) \times \sum_{j=1}^{n} (\gamma_j - \gamma_{j-1})^2\right)$$

with the convention $\gamma_0 = \gamma_n = 0$.

3. One can also use the Gaussian processes machinery: We have seen that $(\mathcal{F}_D, (\cdot,\cdot)_D)$ is a Hilbert space. That means that there exists a Gaussian process $(\tilde{\Gamma}(F))_{F \in \mathcal{F}_D}$ where $F \mapsto \tilde{\Gamma}(F)$ is linear and such that for all $F \in \mathcal{F}_D$, $\tilde{\Gamma}(F) \sim N(0, \|F\|^2)$. This can for instance be achieved by considering an orthogonal basis $(f_1, \dots, f_N)$ of $\mathcal{F}_D$ and to choose i.i.d. standard Gaussian random variables for $\tilde{\Gamma}(f_1), \dots \Gamma(f_N)$. Then, one defines

$$\Gamma(x) := \tilde{\Gamma}(g_{D,x}).$$

Then the vector $(\Gamma(x))_{x \in D}$ is a clearly Gaussian, centered, and the covariance is given by

$$E\left[\Gamma(x)\Gamma(y)\right] = (g_{D,x}, g_{D,y}) = -\sum_z g_{D,x}(z)\Delta g_{D,y}(z) = G_D(x,y).$$

The following remark will be useful later on, and it does in fact provide yet another equivalent construction of the discrete GFF. Fix $x_1 \in D$. Let us first define for all y, $h_{D,x_1}(y) = G_D(x_1, y)/G_D(x_1, x_1)$ (this is the only function on $\mathcal{F}_D$ that is harmonic in $D \setminus \{x_1\}$ and takes the value 1 at x_1). Then, define

$$H_{x_1}(y) = \Gamma(x_1) h_{D,x_1}(y).$$

Note that this is the unique function in $\mathcal{F}_D$ such that $\Delta H(y) = 0$ for all $y \neq x_1$, and $H(x_1) = \Gamma(x_1)$. Define then finally $\hat{\Gamma}(y) = \Gamma(y) - H_{x_1}(y)$.

We note that for all y,

$$E\left[\hat{\Gamma}(y)\Gamma(x_1)\right] = G_D(y, x_1) - h_{D,x_1}(y) G_D(x_1, x_1) = 0$$

so the vector $(\hat{\Gamma}(y), y \in D)$ is independent of $\Gamma(x_1)$ (recall that they are both defined in terms of a single Gaussian vector).

Also for all y, y',

$$E\left[\hat{\Gamma}(y)\hat{\Gamma}(y')\right] = G(y, y') - G(x_1, y)G(x_1, y')/G_D(x_1, x_1).$$

This function is zero as soon as one of the two points y or y' is not in $D \setminus \{x_1\}$. We note also that for all fixed y' this is a function of y such that its Laplacian at y is $-1_{\{y=y'\}}$ for all $y \in D \setminus \{x_1\}$. Hence we deduce that this function is in fact equal to $G_{D\setminus\{x_1\}}(y, y')$. In other words, $\hat{\Gamma}$ is a GFF in $D\setminus\{x_1\}$.

Hence, if we sample independently a Gaussian random variable $\Gamma(x_1) \sim \mathcal{N}(0, G_D(x_1, x_1))$ and $\hat{\Gamma}$ a *GFF* in $D\setminus\{x_1\}$, and then define

$$\Gamma(y) := \hat{\Gamma}(y) + \Gamma(x_1) h_{D,x_1}(y) = \hat{\Gamma}(y) + H_{x_1}(y),$$

we get a GFF in D.

In this way, we in fact first discover the value of the GFF at x_1, and we are left with the task of discovering $\hat{\Gamma}$, which is a GFF in $D \setminus \{x_1\}$. To do this, we can choose x_2 and discover $\hat{\Gamma}(x_2)$, and we are left with the task of discovering a GFF in $D \setminus \{x_1, x_2\}$. We then proceed iteratively, until we have discovered all of the GFF Γ. When $x_1, \ldots, x_N$ are chosen deterministically, one can note that in fact, one is writing $\Gamma(\cdot)$ as $\sum_j \xi_j v_j(\cdot)$ where the ξ_j are i.i.d. Gaussians, and the v_j are some deterministic functions (so we are in fact using an orthonormal decomposition).

However, it is also possible to use the previous construction, when the sequence x_n is itself random. For instance, one is allowed to choose x_2 depending on what one has observed at $\Gamma(x_1)$. In this way, one chooses dynamically to discover the value of Γ at points that depend on the values of Γ at the already observed points. This will give rise naturally to the notion of "local sets" that generalize in some sense the one-dimensional stopping times.

Before turning to the actual definition and study of these local sets, let us state and prove the generalization of the previous result, when one has discovered the value of the GFF on a deterministic finite set $K = \{x_1, \ldots, x_k\}$ of points in D.

Let us first make the following observation concerning the previous decomposition $\Gamma = \hat{\Gamma} + H_{x_1}$: We know that these two functions (in $\mathbb{Z}^2$) on the right-hand side are independent Gaussian vectors,

that the covariance of Γ is $G_D(x,y)$ and that the covariance of $\hat{\Gamma}$ is $G_{D\setminus\{x_1\}}(x,y)$. It follows that the covariance of $(H_{x_1}(y))$ is in fact $G_D(x,y) - G_{D\setminus\{x_1\}}(x,y)$.

Hence, if we iterate this k times, we obtain the following decomposition

$$\Gamma = \Gamma^{K,D} + \Gamma_{K,D},$$

where $\Gamma^{K,D}$ is a GFF in $D \setminus K$, while $\Gamma_{K,D}$ is a Gaussian vector (indexed by $\mathbb{Z}^2$), independent of $\Gamma^{K,D}$ and with covariance function $G_D(x,y) - G_{D\setminus K}(x,y)$.

We can note that almost surely, the function $y \mapsto \Gamma_{K,D}(y)$ is harmonic in $D \setminus K$. This can be either seen from the iterative construction (this function is then almost surely the sum of k harmonic functions) or directly, by checking that $E((\Delta\Gamma_{K,D}(y))^2) = 0$ (which can be read from the covariance function). Hence, the decomposition $\Gamma = \Gamma^{K,D} + \Gamma_{K,D}$ is just the (almost sure) decomposition of $y \mapsto \Gamma(y)$ onto the two orthogonal subspaces $\mathcal{F}_{D\setminus K}$ and $\mathrm{Harm}(D \setminus K, D)$ of $\mathcal{F}_D$.

Note that one can redecompose $\Gamma^{K,D}$ further. For instance, when $K \subset K'$, $\Gamma^{K,D} = \Gamma^{K',D} + (\Gamma^{K,D})_{K'\setminus K,D}$.

1.1.3 Some remarks

1. Let us very briefly mention a further relation between the Green's function and random walk paths. One natural measure on random walk paths in D is the excursion measure μ. This is the measure on the set of nearest-neighbor paths $(Z_n, n \in [0,\tau])$ such that $Z(0) \in \partial D$, $Z(\tau) \in \partial D$ and $Z(0,n) \in D$. These paths are called excursions in D and $\tau = \tau(Z)$ is the length of the excursion. The measure μ assigns to each excursion the mass $(2d)^{-\tau}$. Note that μ is not a probability measure anymore.

   For each excursion Z and each $z \in D$, one can define the time $T_Z(z)$ spent by Z at z; in other words,

   $$T_Z(z) := \sum_{n=1}^{\tau-1} 1_{\{Z_n = z\}}$$

   (we will use a similar notation also for paths that are not excursions). We can note that $\mu(T_Z(z)) = \sum_Z \mu(Z) T_Z(z) = 1$ for all $z \in D$. Because if you take $(Z_n)_{n\in\mathbb{N}}$ is a random walk started from z and stopped at first exit of D we have that

   $$1 = \sum_{\gamma : z \overset{D}{\to} \partial D} P(Z = \gamma) = \sum_{\gamma : z \overset{D}{\to} \partial D} \frac{1}{(2d)^{\#\text{ steps of }\gamma}} = \sum_{\gamma : \partial D \overset{D}{\to} z} \frac{1}{(2d)^{\#\text{ steps of }\gamma}},$$

   so we have that

   $$\begin{aligned}\mu(T_Z(z)) &= \sum_{Z : \partial D \overset{D}{\to} \partial D} \sum_n \mathbf{1}_{\{Z_n = z\}} \frac{1}{(2d)^{\#\text{steps of } Z}} \\ &= \Big(\sum_{\gamma : \partial D \overset{D}{\to} z} \frac{1}{(2d)^{\#\text{steps of }\gamma}} \Big) \times \Big(\sum_{\gamma : z \overset{D}{\to} \partial D} \frac{1}{(2d)^{\#\text{steps of }\gamma}} \Big) = 1\end{aligned}$$

Furthermore,

$$\begin{aligned}\mu(T_Z(z)T_Z(z')) &= \mu\left(\sum_{0<n<\tau} \mathbf{1}_{\{Z_n=z\}}(\sum_{m\le n} \mathbf{1}_{\{Z_m=z'\}} + \sum_{m>n} \mathbf{1}_{\{Z_m=z'\}} - \mathbf{1}_{\{z=z'\}})\right)\\ &= \mu\left(\sum_{0<n<\tau} \mathbf{1}_{\{Z_n=z\}}(2\sum_{n<m<\tau} \mathbf{1}_{\{Z_m=z'\}} - \mathbf{1}_{\{z=z'\}})\right)\\ &= 2\left(\sum_{\gamma:\partial D\to z} \frac{1}{(2d)^{\text{steps of }\gamma}}\right)\left(-\mathbf{1}_{\{z=z'\}} + \sum_{\gamma:z\mapsto\partial D} T_\gamma(z') \times \frac{1}{(2d)^{\#\text{steps of }\gamma}}\right)\\ &= 2G_D(z,z') - \mathbf{1}_{\{z=z'\}}.\end{aligned}$$

From this, it follows immediately that for any real function $u : D \to \mathbb{R}$,

$$2\sum_{x,y} u(x)u(y)G_D(x,y) = \mu((\sum u(x)T_Z(x))^2) + \sum_x u(x)^2 \ge 0$$

which gives another explanation to the fact that G_D is an admissible covariance matrix.

In fact, if we replace in the previous definition the discrete time random walk Z by its continuous-time analog $\tilde{\mu}$ (with exponential waiting times of mean 1 at each site), then for the corresponding excursion measure $\tilde{\mu}$, one has exactly

$$\tilde{\mu}(\tilde{T}_{\tilde{Z}}(z)\tilde{T}_{\tilde{Z}}(z')) = 2G_D(z,z')$$

so that G_D then appears exactly as the covariance function.

For instance, if we define a Poisson point process $(\tilde{Z}^i, i \in I)$ of such excursions in D, with intensity $M\tilde{\mu}/2$, and conditionally on $\bar{Z} := (\tilde{Z}^i, i \in I)$, choose independently a random coin tossing ε_i for each excursion (so that ε_i is ± 1 with probability $1/2$. Let us now define the random field

$$U_M(z) := \sum_i \varepsilon_i \tilde{T}_{\tilde{Z}^i}(z).$$

Then, it is easy to see that (because $E(\varepsilon_i\varepsilon_j|\bar{Z}) = 1_{i=j}$),

$$\begin{aligned}E(U_M(z)U_M(z')) &= E\left(\sum_{i\in I, i'\in I} \varepsilon_i\varepsilon_{i'}\tilde{T}_{\tilde{Z}^i}(z)\tilde{T}_{\tilde{Z}^{i'}}(z')\right)\\ &= E\left(\sum_{i\in I} \tilde{T}_{Z^i}(z)\tilde{T}_{Z^i}(z')\right) = M\tilde{\mu}(\tilde{T}_{\tilde{Z}}(z)\tilde{T}_{\tilde{Z}}(z'))/2 = MG_D(z,z').\end{aligned}$$

Hence, this describes the GFF as the limit when $M \to \infty$ of $(U(z)/\sqrt{M}, z \in D$ (because U_M can be viewed as the sum of M i.i.d. copies of U_1, and one can apply the central limit theorem).

2. Remember that the density of the GFF (in a portion of $\mathbb{Z}^d$) $(\Gamma(x_1), \ldots, \Gamma(x_n))$ at $u_1, \ldots, u_n$ is a constant times

$$\exp\left(-\frac{1}{4(2d)} \times \sum_{x_i\sim x_j} (u_i - u_j)^2\right).$$

Suppose now that the graph D is instead a portion of the triangular lattice in the plane, where the triangles are all equilateral triangles (so the set D is a subset of $T := \mathbb{Z} + e^{i\pi/3}\mathbb{Z}$). Each point has now 6 neighbors in this graph. For each function $F \in \mathcal{F}_D$, we can extend the function F into a continuous function $\overline{F}$ from $\mathbb{R}^2$ onto $\mathbb{R}$ that is linear on each of the triangular faces of T. It is a simple exercise to check that for some simple constant C, and for all $F \in \mathcal{F}_D$, the quantity (F, F) is equal to the integral on the plane of a multiple of $\|\nabla\overline{F}\|^2$ (and that the corresponding scalar product (F_1, F_2) is the integral of $\nabla\overline{F}_1 \cdot \nabla\overline{F}_2$).

This enables in a simple way to couple naturally functions on finer and finer meshes in the triangular grid (say of meshsize 2^{-n}), as a function that is linear on the faces of size 2^{-n} is also linear on the twice larger faces.

1.1.4 Local sets of the discrete GFF

As explained in the previous paragraph, we are now interested in the case where the set K can be random, but defined "dynamically". This leads to the following definition:

Suppose that D is fixed, and that Γ is a GFF in D and $B \subset D$. When there is no ambiguity, we will simply write Γ_B and Γ^B instead of $\Gamma_{B,D}$ and $\Gamma^{B,D}$. We have just seen that for any fixed B, the GFF Γ^B is independent of Γ_B.

When the random set $A \subset D$ is defined on the same probability space as the discrete GFF Γ, we say that **the coupling** (A, Γ) **is local** if for all fixed $B \subset D$, the GFF Γ^B is independent of $(\Gamma_B, \{A = B\})$.

Note that this is a property of the joint distribution of (A, Γ). Sometimes, this property is referred to as saying that "A is a local set of the free field Γ" but we would like to insist here that this does not imply that A is a deterministic function of Γ. For instance, if A is a random set that is independent of Γ, then the coupling (A, Γ) is clearly local.

Here are two instructive examples:

1. Let $D = \{1, 2, \dots, n-1\} \subset \mathbb{Z}$, so that the GFF can be viewed as a Gaussian random walk conditioned to be back at the origin at time n. Choose $x \in \{1, \dots, n-1\}$ randomly and independently of Γ. Then, let

$$y_+ := \max\{y \geq x \ : \ \mathrm{sgn}\Gamma(y) = \mathrm{sgn}\Gamma(x)\} \text{ and } y_- := \min\{y \leq x \ : \ \mathrm{sgn}\Gamma(y) = \Gamma(x)\}$$

   (with the convention $\mathrm{sgn}0 = 0$). In other words, $[y_- - 1, y_+ + 1]$ corresponds to an excursion of Γ in the half-line $\mathbb{R}_+$ or $\mathbb{R}_-$ containing $\Gamma(x)$. It is then a simple exercise to check that $[y_- - 1, y_+ + 1] \cap D$ is local.

2. We can do exactly the same when $D \subseteq \mathbb{Z}^d$: Firs choose x at random independently of Γ, let E denote the largest connected component containing x consisting with points y such that $\mathrm{sgn}\Gamma(y) = \mathrm{sgn}\Gamma(x)$, and then define $A := \overline{E} \cap D$.

Let us also remark that (since one can decompose Γ^B further), if (A, Γ) is a local coupling, then for all $B \subset B'$, the GFF $\Gamma^{B'}$ is independent of $(\Gamma_{B'}, \{A = B\})$ (we will use this fact in the proof of the following lemma). In particular, the GFF $\Gamma^{B'}$ is independent of $(\Gamma_{B'}, A1_{\{\ A \subset B'\}})$.

The following lemma holds:

Lemma 1.1. *Suppose that (A_1, Γ) and (A_2, Γ) are two local couplings (with the same GFF and on the same probability space) such that conditionally on Γ, the sets A_1 and A_2 are independent. Then, $(A_1 \cup A_2, \Gamma)$ is a local coupling.*

It is worthwhile stressing the fact that the conditional independence assumption can not be discarded: Consider for instance the case where $d = 1$, $D = \{-1, 0, 1\}$ and where ξ is a random variable independent of Γ with $P(\xi = 1) = P(\xi = -1) = 1/2$. Then we define $A_1 = \{\xi\}$ and $A_2 = \{\xi \times \mathrm{sgn}(\Gamma(0))\}$. Clearly, A_1 is independent of Γ, and A_2 is independent of Γ, so that (A_1, Γ) and (A_2, Γ) are both local couplings. Yet, $(A_1 \cup A_2, \Gamma(0))$ is not a local coupling (because $\Gamma(0)$ is positive as soon as $A_1 \cup A_2$ has only one element).

If we take $D = \{-2, -1, \dots, 2\}$ and define $A_1 = \{2\xi\}$ et $A_2 = \{\xi \times \mathrm{sgn}(\Gamma(0))\}$, then A_1 and A_2 are two disjoint local sets, but $A_1 \cup A_2$ is not a local set.

Proof. Let U and V denote measurable sets in $\mathbb{R}^D$. Then, for all B_1 and B_2, we let $B = B_1 \cup B_2$, and then (we will again omit the reference to D in the following lines to make the notation simpler)

$$\begin{aligned}
&P\left(\Gamma^B \in U,\ \Gamma_B \in V,\ A_1 = B_1,\ A_2 = B_2\right) \\
&\quad = E\left(P(\Gamma^B \in U,\ \Gamma_B \in V,\ A_1 = B_1,\ A_2 = B_2 \mid \Gamma)\right) \\
&\quad = E\left(\mathbf{1}_{\{\Gamma^B \in U, \Gamma_B \in V\}} P(A_1 = B_1,\ A_2 = B_2 \mid \Gamma)\right) \\
&\quad = E\left(\mathbf{1}_{\{\Gamma^B \in U, \Gamma_B \in V\}} P(A_1 = B_1 \mid \Gamma) P(A_2 = B_2 \mid \Gamma)\right) \\
&\quad = E\left(\mathbf{1}_{\{\Gamma^B \in U, \Gamma_B \in V\}} P(A_1 = B_1 \mid \Gamma) P(A_2 = B_2 \mid \Gamma)\right).
\end{aligned}$$

But we know that Γ^B is independent from $\{\Gamma_B, 1_{A_1 = B_1}\}$ (since $B_1 \subset B$), from which it follows that

$$P(A_1 = B_1 \mid \Gamma) = P(A_1 = B_1 \mid \Gamma_B)$$

is a measurable function of Γ_B, and that the same is true for $P(A_2 = B_2 \mid \Gamma)$. Hence, since Γ_B and Γ^B are independent, it follows that

$$\begin{aligned}
&P\left(\Gamma^B \in U,\ \Gamma_B \in V,\ A_1 = B_1,\ A_2 = B_2\right) \\
&\quad = \ P(\Gamma^B \in U) \times P\left(\Gamma_B \in V,\ A_1 = B_1,\ A_2 = B_2\right)
\end{aligned}$$

If we now fix B and sum over all B_1 and B_2 such that $B_1 \cup B_2 = B$, we conclude that

$$P\left(\Gamma^B \in U,\ \Gamma_B \in V,\ A_1 \cup A_2 = B\right) = P(\Gamma^B \in U) \times P\left(\Gamma_B \in V,\ A_1 \cup A_2 = B\right)$$

which proves that (A, Γ) is a local coupling. □

1.2 Partition functions, loop-soups and spanning trees

We now briefly survey some of the close relations between the discrete GFF, certain measures on random walk loops, and the uniform spanning tree on a graph.

1.2.1 Partition functions

Before turning our attention towards these two latter models, let us first come back to the description of the density of the GFF in $D = \{x_1, \dots, x_n\}$ viewed as a subset of $\mathbb{Z}^d$. As we have already pointed out, the density of $(\Gamma(x_1), \dots, \Gamma(x_n))$ at $u := (u_1, \dots, u_n)$ is equal to

$$\frac{1}{(\sqrt{2\pi})^n \sqrt{\det(G_D)}} \exp(-(u,u)/2),$$

with the convention that for $x_k \notin D$, then $u_k := 0$ (to take into acount the "boundary terms"). Let us define

$$\mathcal{Z}(D) := \sqrt{\det(G_D)} = 1/\sqrt{\det(-\Delta_{\mathcal{F}_D})} = \int_{\mathbb{R}^n} \frac{1}{(\sqrt{2\pi})^n} \exp\left(-\frac{1}{2}(\bar{u},\bar{u})\right) du_1 \dots du_n,$$

where $\bar{u}$ denotes the function in $\mathcal{F}_D$ such that $\bar{u}(x_j) = u_j$.

If we use the previously described decomposition of the GFF, where one first discovers $\Gamma(x_1)$ and is then left to discover the GFF in $D \setminus \{x_1\}$, one immediately gets that (using the fact the $(\bar{u},\bar{u}) = (\bar{u}^{x_1}, \bar{u}^{x_1}) + (\bar{u}_{x_1}, \bar{u}_{x_1})$ and that $\bar{u}_{x_1}(\cdots) = u_1 \times h_{D,x_1}(\cdot)$),

$$\begin{aligned}\mathcal{Z}(D) &= \mathcal{Z}(D \setminus \{x_1\}) \times \int_{\mathbb{R}} \frac{du_1}{\sqrt{2\pi}} \exp\left(-(h,h)u_1^2/2\right) \\ &= \frac{\mathcal{Z}(D \setminus \{x_1\})}{\sqrt{(h,h)}},\end{aligned}$$

where $h = h_{D,x_1}(\cdot) := G_D(x_1,\cdot)/G_D(x_1,x_1)$ is the harmonic function in $D \setminus \{x_1\}$ that is equal to 1 at x_1 and to 0 in the complement of D. Note that because h is harmonic everywhere in D but at x_1,

$$(h,h) = -h(x_1)\Delta h(x_1) = -\Delta h(x_1) = 1/G_D(x_1,x_1).$$

Hence,

$$\mathcal{Z}(D)/\mathcal{Z}(D \setminus \{x_1\}) = \sqrt{G_D(x_1,x_1)},$$

and it follows by induction that if $D = \{x_1, \dots, x_n\}$, then

$$\det(G_D) = \mathcal{Z}(D)^2 := \prod_{j=1}^{n} G_{D\setminus\{x_1,\dots,x_{j-1}\}}(x_j,x_j).$$

This in particularly shows that the product on the right-hand side is in fact independent of the order $x_1, \dots, x_n$ that one did choose for D.

Note that it is in fact very easy to check directly that the product does not depend on the order of the x_j's, for instance by proving the simple identity for all finite A, x and x' in A,

$$G_A(x,x)G_{A\setminus\{x\}}(x',x') = G_A(x',x')G_{A\setminus\{x'\}}(x,x).$$

1.2.2 Wilson's algorithm and erased loops

The previous analysis is closely related to the *Uniform Spanning Tree* (UST in the sequel), and more precisely to the UST in D with wired boundary conditions in ∂D. In order to define this, let us first introduce so notation: One lets $V = V(D, \mathbb{Z}^d)$ denote the graph that one obtains out of $\mathbb{Z}^d$ by identifying all points in the complement of D as one single "outside point". Whenever an edge of $\mathbb{Z}^d$ has at least one of its extremities in D, then it corresponds to exactly one edge in V. The uniform spanning tree in D with wired boundary condition is the uniform measure on the set of subgraphs of V, with no cycle but only one connected component (spanning correspond to the latter property, tree corresponds to the absence of cycles, and uniform corresponds to the uniform measure). Then,

Lemma 1.2. *The number of such spanning trees in $V(D)$ is equal to $(2d)^n / \det(G_D)$.*

It is not surprising that the ratio on the right-hand side is an integer; indeed, one can write it as $(2d)^n \det(-\Delta_{\mathcal{F}_D})$ which is the determinant of an integer-valued matrix.

One instructive way to prove the lemma is to look at the following algorithm (Wilson's algorithm) that constructs a random spanning tree, and to check that it constructs each given spanning tree with the same probability $(2d)^{-n} \det(G_D)$:

We first need the notion of loop-erasure of a path: For any $Z = (z_0, \dots, z_m) \in (\mathbb{Z}^d)^{m+1}$, we define the loop-erasure $L(Z)$ of Z inductively as follows: $L_0 = z_0$, and for all $j \geq 0$, we define inductively $n_j = \max\{n \leq m \ : \ z_n = L_j\}$ and

$$L_{j+1} = Z_{1+n_j}$$

until $j = \sigma$ where $L_\sigma := z_m$. In other words, we have erased the loops of Z in chronological order. The number of steps σ of L is not fixed.

Suppose now that $Z = (Z_n, n \leq \tau)$ is a simple random walk started from $x_1 \in D$, and stopped at its first exit time τ of D, and that $L(Z)$ is its loop-erasure. This is now a nearest-neighbor path joining x_1 to a boundary point of D. For each simple nearest neighbor path $\lambda_1, \dots, \lambda_s$ from x_1 to $x \in \partial D$, when we decompose the probability that $L = \lambda$ according to all possible Z's that give rise to $L(Z) = \lambda$, one immediately gets that

$$P(L(Z) = \lambda) = (2d)^{-s} G_D(\lambda_1, \lambda_1) G_{D\setminus\{\lambda_1\}}(\lambda_2, \lambda_2) \dots G_{D\setminus\{\lambda_1,\dots,\lambda_{s-2}\}}(\lambda_{s-1}, \lambda_{s-1}).$$

The term $(2d)^{-s}$ term corresponds to the jumps of the walk that are still present on the loop-erasure, and the other terms in the product correspond each (for a given j) to the sum of the contributions of all possible paths that the random walk is allowed to perform between $n_j + 1$ and n_{j+1}.

In the sequel, we will also view the loop-erasure of Z as the union of the edges between its successive points, and not just as a collection of points (in order to keep track of a tree-structure).

Now that we know how to define the loop-erased random walk from a point to the boundary of a domain, we are ready to describe Wilson's algorithm to construct a random "tree" in D with wired boundary conditions:

1. We order the n points of D as $x_1, \dots, x_n$.
2. Take as before a simple random walk $(Z_n)_{n\in\mathbb{N}}$ started from x_1 and stopped at its first exist of D, and consider its loop-erasure, that we now call X^1.

3. The interatively, for each $k \in \{2, \dots, n\}$, we do the following: If $y_k \in X^{(k-1)}$, then $X^{(k)} = X^{(k-1)}$, and if $y_k \notin X^{(k-1)}$, then we consider the loop-erasure of a simple random walk started from y_k and stopped at its first exit of $D \backslash X^{(k-1)}$. We then define $X^{(k)}$ to be the union of this loop-erased path with $X^{(k-1)}$.

In this way, $X^{(n)}$ is obviously a tree-like structure (it is a tree if one identifies all points outside of D) and it contains all points of D.

If we are given a given possible outcome T for the tree $X^{(n)}$, then we label the points of D as follows: We call $y_1, \dots, y_{s-1}$ to be the simple path in the tree that goes from x_1 to ∂D (where y_{s-1} is the last point in D in this path). Then, we define y_s to be the next x_j that is not in this already labeled set, and defines $y_s, \dots, y_{s'-1}$ to be the branch in the tree T that joins x_j to $\partial D \cup \{y_1, \dots, y_s\}$, and we iteratively proceed. In this way, we get also an ordering of the vertices of D, that is determined by the tree T.

Inductively, we see that the probability that the tree T is exactly the one constructed by Wilson's algorithm is

$$(2d)^{-n} \prod_{j=1}^{n} G_{D \backslash \{y_1, \dots, y_{j-1}\}}(y_j, y_j).$$

But we have just seen that this quantity is equal to $(2d)^{-n} \det(G_D)$ and does not depend on the order of the points $y_1, \dots, y_n$. It follows readily that the probability that a given spanning tree T is actually chosen by this algorithm does not depend on T (hence, the algorithm samples a uniformly chosen spanning tree) and that the number of spanning trees is exactly $(2d)^n / \det(G_D)$.

1.2.3 Wilson's algorithm and loop-soups

What the previous proof of the fact that Wilson's algorithm samples a uniform spanning tree suggests, is that in some way, the collection of all erased loops during the procedure (or rather the union of all these loops) is independent of the tree that one samples. For example, the previous arguments show that the event that when performing Wilson's algorithm, one erases no loop at all, is independent of the tree, and that its probability is $1/\det G_D$.

A more general statement turns out to hold, and this leads naturally to the definitions of loop-soups ie. to Poisson point processes of unrooted loops in D.

We say that $l = l_0, \dots, l_n$ is a loop in D if it is a nearest-neighbor sequence (in D) such that $l_0 = l_n$. The length n of the loop will be denoted by $|l|$. An unrooted loop $\tilde{l}$ is the equivalence class of loops under circular relabelling i.e. $l_1, \dots, l_n, l_1$ is in the same equivalence class as $l_0, \dots, l_n$.

The unrooted loop-measure μ_D (in D) is basically the measure on the set of unrooted loops in D that assigns to each unrooted loop $\tilde{l}$ the mass $(2d)^{-|\tilde{l}|}$. However, there is a slight catch: In the case where an unrooted loop l consists exactly of the concatenation of J copies of the same unrooted loop (and when J is the maximal such number), then the μ_D mass mass of $\tilde{l}$ will be $(2d)^{-|\tilde{l}|}/J$ (it is divided by the factor J). For instance, an unrooted loop of the type x, y, x, y, x, y, x, y where x and y are neighbors will have mass $(2d)^{-8}/4$.

Let us now consider a Poisson point process of unrooted loops in D, with intensity μ_D. We call this a loop-soup in D. This is a random finite collection $\mathcal{L}$ of unrooted loops $(\lambda_i)_{i \in I}$. Note that if

$D' \subset D$, then μ_D restricted to the set of loops that stay in D' is exactly $\mu_{D'}$. It follows that if one considers the subset of all loops of $\mathcal{L}$ that stay in D', one gets exactly a sample of the loop-soup in D'.

To each unrooted loop λ_j in the loop-soup, and each $x \in D$ that is visited by the loop, we can associate a rooted loop that starts and ends at x, by choosing uniformly at random one of the times of visit of x to be the starting point of the rooted loop. In this way, each loop λ_i is associated to a rooted loop, rooted at each point that it visits. For each given $x \in D$, if we focus on the unrooted loops that go through x, and study the corresponding set of loops that are rooted at x, we get a Poisson point process of rooted loops, with intensity given by the measure that assigns a mass $(2d)^{-|l|}/j$ to each rooted loop l that returns to x exactly j times. Note that this is valid even when the initial loop λ_i was the concatenation of $J \geq 2$ identical copies of a given loop. Indeed, out of the $j = Jk$ possible starting points, only k will give rise to different rooted loops (and there will be J choices of starting points that will give rise to the same rooted loop). Hence in that case, the induced mass of each of the k rooted loops will also be $(2d)^{-|l|}/(Jk) = (2d)^{-|l|}/j$.

Let us now fix $x \in D$ and first evaluate the probability $p(x)$ that no loop of $\mathcal{L}$ visits x. By the definition of Poisson point processes, this probability is $\exp(-\mu(A))$ where A is the set of (unrooted) loops that go through x. Let us define U to be the sum of $(2d)^{-|l|}$ over all rooted loops from x to x in D that do return to x only once. The quantity U^2 therefore counts the sum of $(2d)^{-|l|}$ over all possible (rooted) loops that return to x twice. Taking this into account, we see that

$$\mu(A) = \sum_{j \geq 1} U^j/j = \log(1-U).$$

On the other hand, we know that

$$G_D(x,x) = 1 + \sum_{j \geq 1} U^j = 1/(1-U)$$

from which it follows that and $p(x) = 1/G_D(x,x)$.

Suppose now that there are in fact $r \geq 1$ loops in $\mathcal{L}$ that visits x (which happens with probability $e^{-\mu(A)}\mu(A)^r/r!$). Then, we choose uniformly at random (among all $r!$ choices) an order of these unrooted loops, and for those loops that visit x_1 more than once, say j times, we choose a representative uniformly among the j choices. Finally, we then concatenate all these r loops in order to form one single long loop λ from x to x in D. It is a simple exercise to check that the probability that λ is equal to a given loop $a_0, \ldots, a_k$ is simply $(2d)^{-k}/G_D(x_1,x_1)$. One way to see this is to not that for each fixed $k \geq 1$ (which can be viewed as the number of returns of the loop to x), the sum of $1/(r!j_1 \ldots j_r)$ over all $j_1, \ldots, j_r$ with $j_1 + \cdots + j_r = k$, is equal to 1 (this simple combinatorial fact can be for instance checked by looking at the u^k term in the expansion of $\exp(\log(1-u))$).

Hence, if we now come back to our description of Wilson's algorithm, we see that the previous loop λ (allowing λ to have zero length with probability $1/G_D(x_1,x_1)$) is distributed exactly as the part of the random walk that has been erased until the last visit of x_1 when performing Wilson's algorithm starting at x_1.

Similarly, applying exactly the same reasoning to all steps of Wilson's algorithm (noting that the loop-soup restricted to those loops that do not go through x_1 is exactly a loop-soup in $D \setminus \{x_1\}$), we get the following result:

Proposition 1.3. *The union of all loops that have been erased when performing Wilson's algorithm is distributed exactly (modulo rearrangement, concatenation and relabelling) like the union of all loops in the unrooted loop-soup. Furthermore, this union of loops is independent of the tree that Wilson's algorithm generates.*

1.2.4 Some consequences

Let us now explore some consequences of this relation. Let us consider as before an unrooted loop-soup with intensity μ in D. Let us define for each $x \in D$, $\tau_1(x)$ to be the cumulative time spent at x by all the loops in the loop-soup. We have just seen that $\tau_1(x)$ is a geometric random variable with

$$P(\tau_1(x) = j) = (1-U)U^j \text{ and } E(w^{\tau_1(x)}) = \frac{1-U}{1-Uw},$$

where $U = 1 - (1/G_D(x,x))$. Note that the mean of $\tau_1(x)$ is $U/(1-U) = G_D(x,x) - 1$.

If we now consider an unrooted loop-soup with intensity $\mu/2$ instead of μ and define the cumulative occupation times field $\bar{\tau}_{1/2} := (\tau_{1/2}(x), x \in D)$, we see that $\bar{\tau}_1 = (\tau_1(x), x \in D)$ is equal in distribution to the sum of two independent copies of $\bar{\tau}_{1/2}$ which in turn characterizes the law of $\bar{\tau}$ if one knows that of $\bar{\tau}_1$. For instance, for all positive $w \leq 1$,

$$E(w^{\tau_{1/2}(x)}) = \sqrt{E(w^{\tau_1(x)})} = \sqrt{\frac{1-U}{1-wU}}.$$

More generally, by looking at the properties of the Laplace transforms of the occupation time field of a loop-soup with intensity $c\mu$,

$$\phi_c(u_1, \ldots, u_n) := E\left(\exp\left(-\sum_{j=1}^{n} u_j \tau_c(x_j)\right)\right),$$

and noting that $\phi_{c+c'} = \phi_c \phi_{c'}$ and that $c \mapsto \phi_c$ is decreasing in c for positive u's, we get immediately that

$$\phi_c(u_1, \ldots, u_k) = (\phi_1(u_1, \ldots, u_k))^c.$$

While the relation between the loops erased during Wilson's algorithm and the loop-soup was in fact almost a one-to-one mapping between samples, the relation between these two random objects and the GFF was only pointed out at the level of partition functions. One may nevertheless wonder if there are not hidden more direct relations with the GFF. As we shall now see, this turns out to be indeed the case. Let us first collect a few remarks:

- In our analysis, we have never really used the fact that D was a subgraph of $\mathbb{Z}^d$, beyond the fact that each site in D had the same number $2d$ of neighbors (in $D \cup \partial D$). Hence, all our results hold true when one for instance adds to each site x of D, $d(N-1)$ new edges joining x to itself (this corresponds to the fact that the random walk can stand still). Both ends of these edges are at x, so in this way, one creates a new graph D_N consisting of the points in D, where each point in D now has $2Nd = 2d + 2d(N-1)$ "outgoing" edges (and we keep $\partial D_n = \partial D$). Clearly, when the walker is at x, it will first jump a geometric number of times (with mean N) to itself before finally jumping away to a different point. Hence, if we remember the

definition of the Green's function in terms of mean number of visits to a point by a random walk, we see that this procedure of adding the extra edges multiplies all the values of the Green's function by a factor N i.e. $G_{D_N} = NG_D$. The corresponding $1 - U_N = 1/G_{D_N}(x,x)$ is therefore equal to $1/(NG_D(x,x)) = (1-U)/N$ before adding these stationary edges). The random variable $\tau_1^N(x)$ (the superscript N now refers to the fact that we are looking at the loop-soup in D_N) is still geometric, but with mean $U_N/(1-U_N) = NG_D(x,x) - 1$, so that when $N \to \infty$, the random variable $\tau_1^N(x)/N$ converges in distribution to an exponential random variable with mean $G_D(x,x)$.

We can notice that this shows that in this $N \to \infty$ limit, $\tau_{1/2}^N(x)/N$ converges in distribution to the square of a centered Gaussian random variable with variance $G_D(x,x)/2$. Indeed, the sum of the squares of two i.i.d. Gaussian (with variance $\sigma^2/2$) is well-known to be an exponential random variable (with mean σ). This is the first instance, where we detect that the loop-soups can be naturally related to Gaussian random variables.

This is the first instance of the following general fact:

- Loop-soups with $c = 1$ are nice because they are related to Wilson's algorithm, and nicely amenable to computations involving partition functions and determinants.
- Loop-soups with $c = 1/2$ are nice because they are related in a rather direct way to the Gaussian world. Not surprisingly given the relation between $1/2$ and 1, the formulas involving the loop-soup at $c = 1/2$ (and the GFF) involve square roots of determinants.

- Let us briefly describe the $N \to \infty$ limit of the loop-soup itself. In this limit, the number of "stationary" jumps made by the random walker before each real jump will (when renormalized by factor $1/N$) converge to an exponential random variable with mean 1. In this way, the random walk approaches a continuous time random walk on the graph D with exponential waiting times with mean 1 at each site.

  Note that the discrete loops in D_N on this new graph are allowed to "stand completely still" i.e. to jump from a given point x to itself during their entire life-time. Note that those loops are clearly periodic, and get a mass that is inversely proportional to their length. In the $N \to \infty$ limit (when speeding up the walks by a factor N) one therefore gets, at each point x, a Poisson point process of stationary constant loop, where their length is described by the intensity measure $e^{-t}dt/t$. The e^{-t} terms comes from the fact that the continuous walk will tend to jump away with rate one, and the $1/t$ term comes from the renormalization in the unrooted discrete measure for perdiodic loops.

  If one now focuses on those loops that do not stand still, one can notice that when $N \to \infty$ the probability that two different waiting times (each of them geometric with mean N) are actually equal vanishes as $N \to \infty$. Hence, the contribution of the discrete periodic walks on D_N will vanish in the limit, and one can describe the limiting Poisson point process on continuous loops via its following intensity measure: The mass of each unrooted discrete loop with length $l \geq 2$ on D is $(2d)^{-}l$ (with no renormalizing factor when the discrete loop is periodic) and one adds exponential waiting times at each of its steps.

  One can note that at each point x, there will be infinitely many loops in the loop-soup that stand still (because $\int_0^\infty e^{-t}dt/t = \infty$), but that their cumulative occupation time is almost surely finite (because the expected cumulative time is $\int_0^\infty te^{-t}dt/t = 1 < \infty$). Hence, the cumulative time of the stand-still loop is a random variable with mean 1; it is easy to

see, using analogous arguments as above that in fact the cumulative stand-still time is itself distributed like an exponential random variable of mean 1 when the intensity of the loop-soup is $c = 1$.

The fact that the cumulative occupation time at x of this continuous-time loop-soup with intensity 1 is an exponential random variable corresponds therefore to the fact that it is the sum of the stand-still cumulative time (an exponential random variable) with a geometric number (the number of returns to x in Wilson's algorithm) of i.i.d. exponential random variables with mean one (and this sum is well-known to be exponential itself, just think for instance at jumps of the Poisson process).

In this way, for each $c > 0$, one can define a continuous loop-soup with "intensity" c, and it corresponds indeed to the limit of the $N \to \infty$ limit of the previous discrete speeded-up loop-soup. And for each $x \in D$, one can then define the total cumulative time $\tilde{\tau}_c(x)$ spent at x by this loop-soup.

1.2.5 Loop-soup occupation times field is square of GFF

We have seen above that for each $x \in D$, the law of $\tilde{\tau}_{1/2}(x)$ was the square of a centered Gaussian random variable with variance $G_D(x,x)/2$. In fact, much more is true:

Proposition 1.4. *The field $(\tilde{\tau}_{1/2}(x), x \in D)$ is distributed like the square of a GFF i.e., like $(\Gamma(x)^2/2, x \in D)$ where Γ is a GFF in D.*

The next few paragraphs will outline the proof of this fact.

The idea is to compare the Laplace transforms of $\tilde{\tau}_{1/2}$ and of Γ^2. For $v_1, \dots, v_n$ in $\mathbb{R}_+$, it is easy to evaluate the quantity

$$E\left(\exp\left\{-\sum_{j=1}^{n} v_j \Gamma(x_j)^2/2\right\}\right).$$

Indeed, using the explicit expression of the density of Γ, we see that it is equal to

$$\int_{\mathbb{R}^n} \frac{1}{\sqrt{\det(G_D) \times (2\pi)^n}} \exp\left\{\sum_{i,j} \Delta_D(x_i, x_j) u_i u_j/2 - \sum_{j=1}^{n} v_j u_j^2/2\right\} du_1 \dots du_n,$$

which in turn is simply equal to

$$\sqrt{\det(-\Delta_D)/\det(I_V - \Delta_D)}$$

where I_V denotes the diagonal matrix with diagonal elements $v_1, \dots, v_n$ (here we just use the fact that $(u,u) + \sum_{j=1}^{n} v_j u_j^2$ is positive definite so that the usual change of variable applies).

The goal is therefore to prove that the Laplace transform of $\tilde{\tau}_{1/2}$ is also equal to this quantity, or equivalently (by squaring this in order to express things in terms of the loop-soup with $c = 1$), that

$$E\left(\exp\left\{-\sum_{j=1}^{n} v_j \tilde{\tau}_1(x_j)\right\}\right) = \frac{\det(-\Delta_D)}{\det(I_V - \Delta_D)}.$$

This seems already much more familiar, as we aim at evaluating some additive functional of the collection of loops erased during Wilson's algorithm in terms of determinants involving Green's functions.

The basic idea that leads to this identity goes as follows: One can interpret the quantity

$$\exp\{-\sum_{j=1}^{n} v_j \tilde{\tau}_1(x_j)\}$$

(up to a constant) as the Radon-Nikoym derivative of the loop-soup of a modified continuous Markov process where one introduces a killing with rate v_j at each site x_j with respect to the ordinary continuous loop-soup on D. Then, this expectation is just the ratio of the corresponding partition functions.

Let us first explain how one can treat the case where all v_j's are equal i.e., determine the law of $\tilde{\Sigma} := \sum_x \tilde{\tau}_1(x)$. In our discrete setting, we have seen that it was possible to add "stand-still" edges to our graph, but it is also possible to add "killing edges" to D: For instance, let us fix $k \geq 1$ and to each site x in D, we add k edges that joins it to some point in ∂D.

In this way, on creates a new graph D^k, where each point x in D has now $(2d+k)$ neighbors (in $D \cup \partial D$). When one compares the unrooted loop-soup measure in D with that in D^k, we see that their respective partition functions are $\det G_D$ and $\det G_{D^k}$. More precisely, we see for instance by comparing probabilities of performing exactly the same procedure when sampling Wilson's algorithm in D and in D^k that

$$E\left((2d/(2d+k))^{\sum_x \tau_1(x)}\right) = \frac{\det G_{D^k}}{\det G_D}.$$

We can note that these relations (valid for all k) already characterize the law of $\sum_x \tau_1(x)$.

We are interested in the continuum limit. We can apply the previous relation to D_N instead of D (i.e. D with Nd additional stationary edges at each site) so that

$$E\big((2dN/(2dN+k))^{\sum_x \tau_1^N(x)}\big) = \frac{\det G_{(D_N)^k}}{\det G_{D_N}}.$$

But, as $N \to \infty$,

$$E\big((2dN/(2dN+k))^{\sum_x \tau_1^N(x)}\big) \sim E\big((2dN/(2dN+k))^{N\tilde{\Sigma}}\big) \sim E\left(\exp(-k\tilde{\Sigma}/(2d))\right)$$

while for the right-hand side (just looking at what happens to the mean waiting times at each site), we know that

$$\frac{\det G_{(D_N)^k}}{\det G_{D_N}} \sim (2d/(2d+k))^n \frac{\det G_{D^k}}{\det G_D}.$$

We can recall that G_D is the inverse of $-\Delta_D$, while $-G_{D^k}$ is the inverse of the Laplacian in D^k, and that

$$-(2d+k)\Delta_{D^k} = -2d\Delta_D + kI,$$

so that we can finally conclude that

$$E\left(\exp(-k\tilde{\Sigma}/(2d))\right) = \frac{\det(-\Delta_D)}{\det(-\Delta_D + (kI/2d))}.$$

Hence, the moments of $\exp(-\tilde{\Sigma})$ are the same as those of $\exp(-\sum_x \Gamma(x)^2/2)$, from which we readily deduce the identity in law between $\tilde{\Sigma}$ and $\sum_x \Gamma(x)^2/2$.

In order to control the joint distribution of $(\tilde{\tau}_1(x), x \in D)$ instead of just that of their sum, one can first for instance consider the variant, where the number of killing edges that go out of each site in D can vary (and we add appropriately many stationary edges to each site so that the total number of outgoing edges at each site is the same).

This leads to the expression of

$$E\big(\exp\{-\sum_{j=1}^{n} v_j \tilde{\tau}_1(x_j)\}\big)$$

for enough values v_j (so that this characterizes the joint distribution) and show that it is equal to $\det(-\Delta_D)/\det(I_V - \Delta_D)$. We leave these final details to the reader.

Bibliographical comments

Most results presented in this chapter are rather classical, and they have probably been discovered and re-discovered several times by various mathematicians during the past four decades.

The names of Greg Lawler and Yves Le Jan are naturally associated with our presentation of the relation between the loops erased in Wilson's algorithm and the loop-soups, which is based on our work with Lawler [**?**], and apects of the discrete setting statement Proposition 1.3 has appeared in this discrete setting in [**?**]. Proposition 1.4 is due to Le Jan [**?**].

In the continuous Brownian setting, the intensity of the Brownian loop-soup is usually also denoted by c, but it basically differs from the discrete normalization convention by factor 2 (this is due to the fact that in the continuous case, this value c is interpreted in terms of the central charge of a certain infinite-dimensional Lie algebra for which the convention has the extra factor 2. The loop-soup associated with Wilson's algorithm would have intensity 2 while the loop-soup associated with the Gaussian Free Field would have intensity 1.

Chapter 2

Basics about the continuous two-dimensional GFF

2.1 Definition and first properties

2.1.1 Green's function basics

Let us now quickly derive and list a few properties of the Green's function in two-dimensional domains. Many of these properties have higher d-dimensional counterparts (just replacing $\log 1/|\cdot|$ by $1/|\cdot|^{d-2}$ everywhere), but since the goal of these lectures will be to study the two-dimensional case, we choose to focus on this case already in this chapter.

Of course, the Green's function can be defined and studied in many different ways, and the following probabilistic approach is just one out of many possible ones; it is in fact arguably neither the shortest nor most elegant one (after all, the Green's function is probably a simpler mathematical object than Brownian motion, and we are going to use the latter to define the former). **The reader acquainted to basics of potential theory can safely skip this section.**

Some basic terminology first: We will say that a function f is harmonic on an open subset O of $\mathbb{R}^2$ if it is continuous and if for any closed disc $D(z,\varepsilon)$ contained in O, the mean value of f on the circle $\partial D(z,\varepsilon)$ is equal to $f(z)$. It is well-known that this implies (and equivalent to the fact) that the function is smooth and that its Laplacian vanishes in O.

In the sequel, D shall denote a proper (i.e., $D \neq \mathbb{R}^2$) open subset of the plane, such that all points of ∂D are regular: If $z \in \partial D$, and B denotes a planar Brownian motion started from z, then almost surely, $\inf\{t > 0,\ B_t \notin D\} = 0$. This will for instance hold if D is a finitely connected open subset of $\mathbb{R}^2$, such that all the connected component of $\mathbb{R}^2 \setminus D$ consists of more than one point.

We will denote by P^x (and E^x) the probability measure (and the corresponding expectation) under which B is a Brownian motion started from x, and for all open set O, we denote by τ_O the exit time of O by B. We will simply write τ for τ_D.

We are now ready to define the Green's function $G_D(x,y)$. We are (for fixed $y \in D$) looking for a positive harmonic function in $D \setminus \{y\}$ that vanishes on ∂D. The function $x \mapsto \log(1/|y-x|)$ is harmonic on $D \setminus \{y\}$ and it is positive near x, but does not have the right boundary values on ∂D.

So, we are just going to subtract from this function the harmonic function in D with the same boundary values on ∂D. In other words, When $y \in D$ and $x \in D \setminus \{y\}$, we set

$$G_D(x,y) := \frac{1}{2\pi} \log \frac{1}{|x-y|} - \frac{1}{2\pi} E^x \left(\log \frac{1}{|B_\tau - y|} \right).$$

When either x or y are not in D, we set $G_D(x,y) = 0$.

For each fixed $y \in D$, the function $x \mapsto G_D(x,y)$ has the following properties:

- It is harmonic on $D \setminus \{y\}$: This is just $x \mapsto \log|x-y|$ is harmonic, and the second term is also harmonic because of the strong Markov property of Brownian motion (at the hitting times of circles).
- It tends to 0 on ∂D: When $x_n \to x_\infty \in \partial D$, then $G_D(x_n, y) \to 0$. This is a direct consequence of the regularity of the boundary. Indeed, for each $\varepsilon > 0$, with a probability that tends to 1 as $n \to \infty$, the Brownian motion that starts from x_n will exit D in an ε-neighborhood of x.
- The mapping $x \mapsto G_D(x,y) - (2\pi)^{-1} \log(1/|x-y|)$ extends continuously to the point y, and the extension is therefore also harmonic at y.

The maximum principle shows immediately that the previous three properties characterize uniquely the function $x \mapsto G_D(x,y)$ (if one considers another such function, then difference would be harmonic in D and vanishes on ∂D). This shows also readily that the Green's function is positive, and that (for fixed x and y), it is an increasing function of the domain D (the larger D, the larger the Greens' function).

We can also note that if D_n is an increasing sequence of open sets such that $\cup D_n = D$, then for all $x \neq y$ in D,

$$G_{D_n}(x,y) \to G_D(x,y)$$

as $n \to \infty$ by dominated convergence (just notice that $\tau_{D_n} \to \tau$ almost surely). For instance, and we shall be using the following notation throughout this section, one can first define A_n to be the union of all closed 2^{-n}-dyadic squares (by this we mean squares of the type $S^n_{j,l} := [j2^{-n}, (j+1)2^{-n}] \times [l2^{-n}, (l+1)2^{-n}]$) that do intersect the complement of D, and then define $D_n := D \setminus A_n$.

If one fixes x and lets y vary, it is immediate to check that $y \mapsto G_{D_n}(x,y)$ (for the previously defined D_n) is a function on $D_n \setminus \{x\}$ that is harmonic (as mean value of harmonic functions) and that behaves like $(2\pi)^{-1} \log 1/|y-x|$ as $y \to x$. Furthermore, it vanishes as $y \to y_\infty \in \partial D_n$ (using the fact that the probability that Brownian motion started from x exits D_n in an ε-neighborhood of y_∞ decreases at least like a constant times $\varepsilon^{1/2}$), so that in fact,

$$G_{D_n}(x,y) = G_{D_n}(y,x)$$

for all $x \neq y$ in D_n, and letting $n \to \infty$, it finally follows that $G_D(x,y) = G_D(y,x)$ for all $x \neq y$ in D.

We can now list a few further properties:

- Let us just do a trivial observation in order to try to prevent us from doing normalization errors. When D is the unit disc, then $G_D(x,y) = (2\pi)^{-1} \log(1/|x-y|)$, so that in particular $\int_D G_D(0,y)dy = \int_0^1 r \log(1/r) dr = 1/4$.

- When f is a continuous function that vanishes on the boundary of D, let us define

$$F(x) = \int_D f(y) G_D(x,y) dy,$$

which is easily shown to be a continuous function that vanishes on the boundary D. The function $x \mapsto \int_D f(y) E^x(\log 1/|B_\tau - y|) dy$ is clearly harmonic in D (as mean of harmonic functions).

If we fix z and subtract $F(z)$ from the mean value of F on a small circle of radius ε around z, it is therefore the same as when one replace F by $x \mapsto \int_D (2\pi)^{-1} f(y) \log(1/|x-y|) dy$, so that we see readily that it is equivalent to $f(z)\varepsilon^2/4$ when $\varepsilon \to 0$, from which it follows easily that F is C^2 and that $-\Delta F = f$. So, one can interpret the kernel G_D as the inverse of $-\Delta$ as in the discrete case (if defined on appropriate function spaces).

Note also that this shows that

$$\int_{D\times D} f(x) G_D(x,y) f(y) dx dy = -\int_D F(x) \Delta F(x) dx = \int_D |\nabla F(x)|^2 dx \geq 0.$$

Also, we can note easily that the function

$$x \mapsto E^x(\int_O^\tau f(B_t) dt)/2$$

is continuous, that its Laplacian is equal to f (by looking at circular averages, here one can note that the expected time that a d-dimensional takes to leave a ball of radius ε when started from its center is ε^2/d, and that for a smooth function F, the mean value of its value on a circle of radius ε minus its value at the center is $\varepsilon^2/(2d)$) and that its boundary value vanish. Hence the maximum principle shows that

$$E^x(\int_O^\tau f(B_t) dt/2) = \int_D f(y) G_D(x,y) dy.$$

By dominated converges, letting f converge to the indicator function of an open subset A of D, we see that

$$E^x(\int_O^\tau 1_{B_t \in A} dt) = 2\int_A G_D(x,y) dy.$$

This is indeed the continuous analogue of the definition of the discrete Green's function as mean number of visits of a point by a random walk started from another point (mind however the usual factor 2 that appears because in the continuum, the more natural operator associated to Brownian motion is in fact $\Delta/2$, which in turn is the inverse of $2G_D$). In fact, if we consider A to be the disc of radius ε around y, we see that

$$2G_D(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} E_x(\int_0^\tau 1_{|B_t - z| < \varepsilon} dt).$$

- In fact (but we will not need it here), the previous analysis can be pushed a little further, by choosing an orthonormal (in L^2) basis φ_j of eigenfunctions of the Hilbert Schmidt operator $-\Delta$ (associated to the eigenvalues λ_j), which implies readily that

$$G_D(x,y) = \sum_j \frac{1}{\lambda_j} \varphi_j(x) \varphi_j(y).$$

It is also possible and not very difficult to see that for each t, the law of $B_{\min(t,\tau)}$ has a density $p_{D,t}(x,y)$ in D that is jointly continuous in t,x,y on $(0,\infty)\times D\times D$, so that

$$2G_D(x,y)=\int_0^\infty p_{D,t}(x,y)dt.$$

- The characterization of $G_D(x,\cdot)$ as the unique harmonic function in $D\setminus\{x\}$ with prescribed boundary conditions implies immediately that when Φ is a conformal transformation from D onto $\Phi(D)$, then
$$G_{\Phi(D)}(\Phi(x),\Phi(y))=G_D(x,y).$$
By conformal invariance and using the expression of the Green's function in the unit disc, this provides a definition of the Green's function in terms of a conformal map Φ from D onto $\mathbb{U}$ when D is simply connected.

- The definition shows that as $y\to x$,
$$G_D(x,y):=\frac{1}{2\pi}\log\frac{1}{|x-y|}-\frac{1}{2\pi}E^x\left(\log\frac{1}{|B_\tau-x|}\right)+o(1).$$
Suppose now that D is simply connected and that Φ is the conformal transformation of the unit disc onto D with $\Phi(0)=x$ and $\Phi'(0)\in\mathbb{R}_+$. This derivative is sometimes called the conformal radius of D at x. Then, using the explicit expression of $G_{\mathbb{U}}$, we get that the constant term in the expansion of $G_D(x,y)$ as $y\to x$ is equal to $(2\pi)^{-1}\log\Phi'(0)$.

- It is also a rather simple exercise to check that discrete Green's functions in a discrete fine-mesh approximation of a domain D converge to continuous Green's functions when the mesh-size goes to 0 (just mind the factor 2). This is not surprising in view of the fact that discrete random walks converge to Brownian motion.

- Suppose now that $D'\subset D$, and that as for D, all boundary points of D' are regular. Define for any $x\neq y$ in D,
$$H_{D,D'}(x,y)=G_D(x,y)-G_{D'}(x,y)$$
Note that this function is equal to $G_D(x,y)$ as soon as either x or y are not in D', and that it can be extended continuously to the points where $x=y\in D'$.

  Clearly, when $x\in D'\subset D$, the function $y\mapsto H_{D,D'}(x,y)$ is harmonic in $D'\setminus\{x\}$ and continuous at x (and therefore also harmonic at x). Hence, one can easily check that it is the (unique) harmonic extension in D' of the function that is equal to $G_D(x,y)$ for all $y\in\partial D'$.

  We can also notice that for $A\subset D'$, one has
$$\int_A(G_D(x,y)-G_{D'}(x,y))dy=E^x\left(\int_{\tau'}^{\tau}1_{B_t\in A}dt\right)$$
so that one interpret the quantity $G_D(x,y)-G_{D'}(x,y)$ as the density of the cumulative occupation time in the neighborhood of y by Brownian motion started from x, but restricted to those times between τ' and τ.

  If $\nu_{x,D'}$ denotes the law of $B_{\tau'}$, we see that in fact that for $x,y\in D'$,
$$G_D(x,y)-G_{D'}(x,y)=\int d\nu_{x,D'}(dz)d\nu_{y,D'}(dz')G_D(z,z'),$$

and that (letting $y \to x$), the quantity

$$\int d\nu_{x,D'}(dz) d\nu_{x,D'}(dz') G_D(z,z')$$

is finite when $x \in D'$.

- Finally, let us briefly remark (and leave this as a simple exercise) that if μ is a compactly supported positive measure in D such that

$$\int_D G_D(x,y) d\mu(x) d\mu(y) < \infty,$$

and if for all $\varepsilon > 0$, one defines μ^ε to be the convolution with a well-chosen smooth positive test function supported on the disc of radius ε around the origin, then by dominated convergence, one can readily conclude that

$$\int_D G_D(x,y) d\mu^\varepsilon(x) d\mu^\varepsilon(y) \to \int_D G_D(x,y) d\mu(x) d\mu(y).$$

Similarly, with obvious notation,

$$\int_D G_D(x,y)(d\mu^\varepsilon(x) - d\mu(x))(d\mu^\varepsilon(y) - d\mu(y)) \to 0.$$

One can also for each n subdivide D into dyadic squares of the type $C_{j,l} := [j2^{-n}, (j+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n})$ and define the measure μ_n as $\sum_{j,l} \mu(C_{j,l}) \lambda_{j,l}$ where $\lambda_{j,l}$ is now the uniform (i.e. multiple of the Lebesgue measure) probability measure on $C_{j,l}$. Then, as before, it is easy to check that

$$\int_D G_D(x,y)(d\mu_n(x) - d\mu(x))(d\mu_n(y) - d\mu(y)) \to 0$$

as n tends to infinity.

2.1.2 GFF definition

Disclaimer. As in the discrete case, there are several possible ways to view and define the GFF. We choose here to use the concrete and somewhat pedestrian one as a random process, that has the advantage of explicitly pointing out the regularity features of this random field.

Recall some extremely elementary facts from measure theory: A random real-valued process indexed by some set $\mathcal{A}$ is just a collection of random variables $(X_a, a \in \mathcal{A})$ defined on the same probability space. The *law of the process* is a measure on $\mathbb{R}^{\mathcal{A}}$ (endowed with the product σ-field) is characterized by its finite-dimensional distributions (i.e. the knowledge of the law of the finite-dimensional vector $(X(a_1), \ldots, X(a_n))$ for each $a_1, \ldots, a_n$). Conversely, if one is given a family of finite-dimensional distributions that is compatible (taking the marginal distribution of one of these distributions, one gets the corresponding new marginal distribution), then it is possible (this is Kolmogorov's extension theorem) to construct a probability space and process $(X_a, a \in \mathcal{A})$ on this probability space, with these finite-dimensional distributions. In this setting, one can note that it

is a priori not possible to "observe simultaneously" more than a countable collection of variables X_a by definition of the product σ-field, and in order to make sense of things such as stopping times etc., it is convenient to construct "versions" of the process where for instance, on a set of probability one, it turns out that $a \mapsto X_a$ is continuous (at some point, or in some given subset of $\mathcal{A}$). This will be the type of very pedestrian route that we will use to define the GFF.

Throughout this section, D will be fixed with the same conditions as before (D is open with a regular boundary), but we will also assume that it is a subset of the unit square (which is actually not a restriction, because of conformal invariance).

We define the set $\mathcal{M}^+$ of finite measures supported in D such that

$$\int_D G_D(x,y)d\mu(x)d\mu(y) < \infty.$$

We also define the vector space $\mathcal{M}$ of signed measures $\mu^+ - \mu^-$ where μ^+ and μ^- are in $\mathcal{M}^+$.

We say that the process $(\Gamma(\mu), \mu \in \mathcal{M})$ is a Gaussian Free Field in D if it is a centered Gaussian process with covariance function

$$E(\Gamma(\mu)\Gamma(\nu)) = \int_{D\times D} G_D(x,y)d\mu(x)d\nu(y).$$

Recall that the law of a random process is just defined by its finite-dimensional marginals (which is what this definition provides for Γ.

We can also note that for all $\lambda \in \mathbb{R}$ and all μ and ν, one has almost surely

$$\Gamma(\lambda\mu) = \lambda\Gamma(\mu) \text{ and } \Gamma(\mu+\nu) = \Gamma(\mu) + \Gamma(\nu)$$

(just noticing that the second moment of the differences vanish). It follows that the distribution of the process is characterized by this linearity relation and the fact that for each μ in $\mathcal{M}$, $\Gamma(\mu)$ is a centered Gaussian random variable with variance $\int G_D(x,y)d\mu(x)d\mu(y)$.

Mind (as always for random process indexed by fairly large sets) the order of the a.s. and the for all μ. We would like already to stress that for general *random* measures $\mu \in \mathcal{M}$ (coupled to the GFF), $\Gamma(\mu)$ is not necessarily a well-defined random variable i.e. it is not necessarily measurable.

Before starting to work with the GFF, one needs to check that it is indeed well-defined i.e. that the formula that we gave for the variance of $\Gamma(\mu)$ is always non-negative (by the theory of Gaussian random vectors, this is a sufficient condition). This is very easily checked (for instance, just use the approximation scheme of μ by a smooth $\mu^\varepsilon(dz) = f^\varepsilon(z)dz$ and the fact that $\int_{D\times D} f(x)f(y)G_D(x,y)dxdy \geq 0$ for smooth functions).

Now, we can make a few further comments.

1. Suppose that one knows the countable collection of random variables $(\Gamma(\mu^n_{j,l}))$, where $\mu^n_{j,l}$ is the uniform distribution in the dyadic square $S^n_{j,l}$, then using the last remark of the previous subsection, we know that one can find a sequence μ_n of linear combinations of the $\mu^n_{j,l}$'s so that $\Gamma(\mu_n)$ converges in L^2 to $\Gamma(\mu)$ (and therefore choosing some appropriate deterministic subsequence, we get almost sure convergence). Hence, the knowledge of all these $(\Gamma(\mu^n_{j,l}))$'s enables to recover each $\Gamma(\mu)$ individually (i.e., for each μ, one can almost surely recover $\Gamma(\mu)$).

In the sequel, an important role will be played by the σ-field $\mathcal{F}_A$ generated by all $\Gamma(\mu)$, for those $\mu \in \mathcal{M}$ that are supported in some compact set A.

2. **Circular averages are particularly nice.** Let $\lambda_{z,r}$ denote the uniform distribution on the circle of radius r around z. Suppose that z_0 is fixed and that $r_0 < d(z_0, \partial D)$. We define, for all $r \leq r_0$, the circular average

$$\gamma(z_0, r) = \Gamma(\lambda_{z_0,r}).$$

Suppose also that μ is a probability measure in D that is supported in $D \setminus D(z_0, r_0)$. Then, we observe that for $r \leq r_0$,

$$E(\gamma(z_0, r)\Gamma(\mu)) = \int d\mu(x) G_D(x, y) d\lambda_{z_0,r}(dy) = \int d\mu(x) G_D(x, z_0)$$

because of the harmonicity properties of the Green's function. Hence, we see that

$$E((\gamma(z_0, r) - \gamma(z_0, r_0))\Gamma(\mu)) = 0.$$

The process $r \mapsto \gamma(z_0, r) - \gamma(z_0, r_0)$ defined for all $r \in (0, r_0]$ is therefore independent of all the $\Gamma(\nu)$'s for ν supported outside of $D(z_0, r_0)$

Similarly, using the harmonicity of the Green's function, we get readily that for all $r < r' \leq r_0$,

$$\begin{aligned} E((\gamma(z_0, r) - \gamma(z_0, r'))^2) &= \int d\lambda_{z_0,r}(x) G_D(x, z_0) - \int d\lambda_{z_0,r'}(x) G_D(x, z_0) \\ &= \int d\lambda_{z_0,r}(x) G_{D(z_0,e')}(x, z_0) = \log(r'/r). \end{aligned}$$

It follows (recall the properties relating independence and covariance for Gaussian processes) that the process $b_{z_0,r_0} := (\gamma(z_0, r_0 e^{-u}) - \gamma(z_0, r_0), u \geq 0)$ is a one-dimensional Brownian motion, independent of the σ-field generated by all $\Gamma(\mu)$'s with μ supported outside of $D(z_0, r_0)$.

Note that this shows in particular that if we are given a countable collection of disjoint open disks $D(z_1, r_1), \ldots$ in D, then the collection of processes b_{z_j,r_j} are independent Brownian motion.

3. **Conformal invariance of the GFF.** The GFF inherits conformal invariance properties from the conformal invariance of the Green's function. Suppose that D and D' are two conformally equivalent domains in the plane (i.e. there exists an angÃ©e-preserving bijection Φ from D onto D'). Then, we have seen that $G_D(x, y) = G_{D'}(\Phi(x), \Phi(y))$. Hence, if the GFF would have been an actual function, then the law of this function would have been conformally invariant. Here, it is conformally invariant "as a generalized function". In other words, it means for instance that $\Gamma_D(\mu)$ is distributed like $(\Gamma_{D'}(\mu')$, where μ' is the measure defined by

$$\mu'(\Phi(A)) := \int_A d^2x \mu(dx) |\Phi'(x)|^2.$$

This conformal invariance property will enable us to restrict ourselves to the case where D are bounded (and subsets of the unit square say).

2.1.3 Regularity and orthogonal decompositions

One can use Kolmogorov's criterion in order to prove the existence of continuous versions or modifications of a given stochastic process indexed by a subset of an d-dimensional space: If for some process $X = (X_u, u \in U)$ indexed by $u \in U \subset [0,1]^d$, there exists a even integer $2n$ and positive positive δ and C so that for all u, u',

$$E((X(u) - X(u'))^{2n}) \leq C|u - u'|^{d+\delta},$$

then there exists a modification of X such that on a set of probability one, the map $u \mapsto X(u)$ is continuous on U.

Recall also the following trivial fact: When X is a centered Gaussian random variable with variance σ^2, then (for some universal constant c_n), one has $E(X^{2n}) = c_n \sigma^{2n}$.

It follows immediately that if for some Gaussian process $(X_u, u \in U)$ indexed by $u \in U \subset [0,1]^d$, there exists positive ε and C so that for all u, u',

$$E((X(u) - X(u'))^2) \leq C|u - v|^{\varepsilon},$$

then there exists a modification of X such that on a set of probability one, the map $u \mapsto X(u)$ is continuous on U (just use Kolmogorov's criterion for n so that $n\varepsilon > d$).

It is then a simple exercise to apply this for various finite-dimensional subsets of the space of measures $\mathcal{M}$:

1. There exists a version of the process $(z, r) \mapsto \gamma(z, r)$ that is continuous on $\{(z, r) \in D \times (0, \infty),\ r < d(z, \partial D)\}$. This shows for instance that it is possible to construct a modification of the process Γ such that almost surely all the Brownian motions $b_{z,r}$ are (simultaneously) continuous.

2. If $\bar{\gamma}(z, r)$ now denote the value of Γ on the Lebesgue measure in the disc of radius r around z, we also get that it is possible to find a version of this process that is jointly continuous in z and r (and this time $r = 0$ is allowed). Note that $r \mapsto \bar{\gamma}(z, r)$ is in fact differentiable and that its derivative is related to the circular averages.

3. Let us fix A to be a finite union of closed dyadic squares in D, and define $O := D \setminus A$. For each x in D, define as before $\nu_{O,x}$ to be the exit distribution of O by Brownian motion started from x. The process $(\Gamma(\nu_{O,x}), x \in O)$ is now indexed by a subset of D, and it is again easy to check that
$$E((\Gamma(\nu_{O,x}) - \Gamma(\nu_{O,x'}))^2) \leq C_\varepsilon |x - x'|$$
for some constant C_ε that is uniform over all choices of x and x' that are at distance greater than ε from ∂O. It therefore follows that one can find a version of this process so that $x \mapsto \Gamma(\nu_{O,x})$ is continuous in O (on a set of full probability measure). Note that the definition shows also immediately that this continuous function h_A is harmonic in O, and that its covariance function for $x, y \in O$, is $G_D(x, y) - G_O(x, y)$.

   When μ is now any measure in $\mathcal{M}$, we define

$$\Gamma_A(\mu) = \Gamma(\mu 1_A) + \int_{D \setminus A} h_A(x) \mu(dx).$$

This process is a functional of Γ, it is Gaussian and its covariance function is given by

$$E(\Gamma_A(\mu)\Gamma_A(\mu')) = \int_{D\times D} (G_D(x,y) - G_O(x,y))\mu(dx)\mu(dx').$$

Intuitively, it is nothing else that Γ when restricted to measures supported on A, and it is harmonically extended to $O = D \setminus A$. Note that $\Gamma_A(\mu)$ is in fact equal to some $\Gamma(\bar{\mu})$ with $\bar{\mu}$ supported in A. The process Γ_A is therefore $\mathcal{F}_A$ measurable. Clearly, we also know that $\mathcal{F}_A \subset \sigma(\Gamma_A)$ (because $\Gamma(\mu) = \Gamma_A(\mu)$ for measures μ supported in A), so that equality holds.

We now define

$$\Gamma^A := \Gamma - \Gamma^A.$$

Using again the harmonicity properties of the Green's function we see that the processes Γ^A and Γ_A are independent. Indeed, note first that Γ^A vanishes on all measure supported in A. Then, note that for all ν supported in A and all μ supported in O,

$$E(\Gamma(\nu)\Gamma(\mu)) = E(\Gamma(\nu)h_A(\mu))$$

which implies the independence between $\Gamma(\nu)$ and $\Gamma^A(\mu)$, and therefore between $\Gamma^A(\mu)$ and $\mathcal{F}_A$.

The covariance structure of Γ^A is therefore given by the difference between that of Γ and that of Γ_A:

$$E(\Gamma^A(\mu)^2) = \int\int d\mu(x)d\mu(y)G_O(x,y).$$

Γ^A is therefore a GFF in O that is independent of Γ_A. (this $\Gamma = \Gamma_A + \Gamma^A$ decomposition corresponds of course to some orthogonal decomposition in a Hilbert space). In fact, we have just proved that for all $\mu \in \mathcal{M}$,

$$\Gamma_A(\mu) = E(\Gamma(\mu)|\mathcal{F}_A)$$

almost surely.

4. Suppose now the A is just as before, that A' is another such union of dyadic squares and that $A \subset A'$. Then it is a simple exercise that we safely leave to the reader to check that (almost surely)

$$\Gamma^{A'} = (\Gamma^A)^{A'} \text{ and } \Gamma_{A'} = \Gamma_A + (\Gamma^A)_{A'},$$

ie., that Γ can be decomposed as follows into the sum of the following three independent Gaussian processes

$$\Gamma = \Gamma_A + (\Gamma^A)_{A'} + \Gamma^{A'}.$$

5. We will come back to this in more detail a little later and in a more general setting, but let us already mention that the previous facts concerning unions of closed dyadic squares can be generalized to general compact sets. Suppose that O is any given open subset of D with a regular boundary and let A denote its complement. Then, define its 2^{-n}-dyadic approximation O_n as in the Green's function section (and the corresponding complements A_n that are defined as the union of all closed 2^{-n} dyadic squares that intersect A). Then, we see that for all μ,

$$\Gamma_{A_{n+1}}(\mu) = E(\Gamma_{A_n}(\mu)|\mathcal{F}_{A_{n+1}})$$

almost surey, which means that $\Gamma_{A_n}(\mu)$ is an inverse martingale in L^2 (mind that the σ-fields are decreasing in n). It therefore converges almost surely and in L^2 as $n \to \infty$. Actually (just noting that $\Gamma(\mu)$ is in L^p for all $p \geq 2$) it converges in any L^p space.

We can then *define* the limit to be $\Gamma_A(\mu)$ and then

$$\Gamma^A(\mu) = \Gamma(\mu) - \Gamma_A(\mu).$$

It is then a simple exercise to check that indeed Γ_A and Γ^A are independent Gaussian processes with covariance structure given by G_O and $G_D - G_O$ respectively (as limit of Gaussian processes), and then finally that there exists a version of the process, such that Γ_A when restricted to O is a harmonic function h_A (using Kolmogorov's criterion just as when A was a usion of dyadic squares).

It is also immediate to check that in the special case where A is a union of dyadic squares, this definition of Γ_A (as limit of Γ_{A_n}'s) coincides with the previous one.

2.2 Local sets

2.2.1 Definition and first properties

We are now ready to define *local couplings of random sets to the continuous GFF* (or in short, local sets of the GFF). Our random sets will be random compact subsets of $\overline{D}$ (this set of compact sets is endowed with the usual Hausdorff metric).

We are going to define local sets in two stages: First we will define a notion of local sets for random unions of 2^{-n}-dyadic squares, and then we will define (by approximation) the general notion of local sets. We will use the following notation in the present section: Finite unions of closed dyadic squares will be denoted by small letters a, b etc, and we will be able to apply the results of the previous section (construction of Γ_a etc.) to those. Capital letters A will be used to denote *random* closed subsets of the unit square.

Dyadic local sets: Suppose that $D \subset [0,1]^2$, and that $n \geq 1$ is fixed. We say that the random compact set $A \subset [0,1]^d$ (defined on the same probability space as the GFF Γ) is a 2^{-n}-dyadic local set if:

- It is a random union of closed 2^{-n}-dyadic squares (it can therefore take only finitely many values).
- For all deterministic union a of closed 2^{-n}-dyadic squares, the GFF Γ^a (in $D \setminus a$) is independent of $\sigma(\mathcal{F}_a, \{A = a\}) = \sigma(\Gamma_a, \{A = a\})$.

This definition is of course very much reminiscent of the discrete GFF setting. Mind that (as in the discrete setting), this is a property of the joint distribution of (A, Γ). Exactly as in the discrete setting, one can easily prove that

1. If A is a random union of 2^{-n} dyadic sets that is independent of Γ, then it is a 2^{-n}-dyadic local set (of course those are not particularly interesting....).

2. It is possible to replace in the last line of this definition $\sigma(\mathcal{F}_a, \{A = a\})$ by $\sigma(\mathcal{F}_a, \{A \subset a\})$, using the fact that $\{A \subset a\} = \cup_{a'} \{A = a'\}$ and recalling that for $a' \subset a$, one can decompose $\Gamma^{a'}$ into the sum of the two independent processes Γ^a and $(\Gamma^{a'})_a$.

3. When A and A' are two 2^{-n}-dyadic local sets coupled with the same GFF Γ that are conditionally independent given Γ, then $A \cup A'$ is also a 2^{-n}-dyadic local set coupled with Γ. The proof is almost a paste-and-copy of the discrete case:

   For any given $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_{m'}$ and open sets $U_1, \dots, U_m, V_1, \dots, V_{m'}$ in $\mathbb{R}$, we define the $\sigma(\Gamma^b)$ and $\sigma(\Gamma_b)$ measurable events

$$U^b = \{\forall j \leq m,\ \Gamma^b(\mu_j) \in U_j\} \text{ and } V_b = \{\forall j \leq m',\ \Gamma_b(\nu_j) \in V_j\}.$$

   Note that the set of events U^b is stable under finite intersections and generates $\sigma(\Gamma^b)$, and that the family of events V_b and $V_b \cap \{A = a\}$ is stable under finite intersections and generates $\sigma(\Gamma_b, \{A = a\})$ (we can also make a similar remark replacing A by A'). Then, for all a and a', we let $b = a \cup a'$, and check that

$$\begin{aligned} &P\left(U^b,\ V_b,\ A = a,\ A' = a'\right) \\ &\quad = E\left(P(U^b,\ V_b,\ A = a,\ A' = a' \mid \Gamma)\right) \\ &\quad = E\left(1_{U^b, V_b} P(A = a,\ A' = a' \mid \Gamma)\right) \\ &\quad = E\left(1_{U^b} 1_{V_b} P(A = a \mid \Gamma) P(A' = a' \mid \Gamma)\right). \end{aligned}$$

   But we know that Γ^b is independent from $\sigma(\Gamma_b, 1_{A=a})$ (since $a \subset b$), from which it follows that

$$P(A = a \mid \Gamma) = P(A = a \mid \Gamma_b)$$

   is measurable with respect to $\sigma(\Gamma_b)$, and that the same is true for $P(A' = a' \mid \Gamma)$. Hence, since Γ_b and Γ^b are independent, it follows that

$$\begin{aligned} &P\left(U^b,\ V_b,\ A = a,\ A' = a'\right) \\ &\quad = \ P(U^b) \times E\left(1_{V_b} P(A = a \mid \Gamma) P(A' = a' \mid \Gamma)\right) \\ &\quad = \ P(U^b) \times P\left(V_b,\ A = a,\ A' = a'\right) \end{aligned}$$

   If we now fix b and sum over all a and a' such that $a \cup a' = b$, we conclude that

$$P\left(U^b,\ V_b,\ A \cup A' = b\right) = P(U^b) \times P\left(V_b,\ A \cup A' = b\right)$$

   from which we deduce that $A \cup A'$ is a 2^{-n}-dyadic local set.

Given that a 2^{-n}-dyadic local set A can take only finitely many values, we can define without any problem

$$\Gamma_A = \sum_a 1_{A=a} \Gamma_a,\ \Gamma^A = \sum_a 1_{A=a} \Gamma^a,\ h_A = \sum_a 1_{A=a} h_a.$$

and note that with probability one, h_A is in fact a random harmonic function in the random set $D \setminus A$. Furthermore, conditionally on the random set A (and on Γ_A), Γ^A is a GFF in $D \setminus A$.

One can also define the σ-field $\mathcal{F}_A$ as the σ-field of all events U such that for all a, $U \cap \{A = a\} \in \mathcal{F}_a$. In other words, this is the set of events that can be decomposed as $\cup(\{A = a\} \cap U_a)$ with $U_a \in \mathcal{F}_a$. It is the immediate that $\Gamma_A(\mu) = \sum_a 1_{A=a} \Gamma_a$ is $\mathcal{F}_A$ measurable, and that $\Gamma_A(\mu) = E(\Gamma(\mu) | \mathcal{F}_A)$ almost surely.

General local sets. We now finally define general local couplings. For all compact subset A of $[0,1]^2$, we define its 2^{-n}-dyadic approximation A_n as before (it is the union of all closed 2^{-n} dyadic squares that intersect A), and we let $O_n = D \setminus A_n$. Let A be a random compact set A defined on the same probability space as the GFF Γ.

We say that A is locally coupled to Γ (or equivalently that it is a local set of the GFF) if for any $n \geq 1$, the set A_n is a 2^{-n}-dyadic local set.

A not very exciting example of local sets are random compact sets that are independent of Γ. We can now list some properties of local sets:

1. If A and A' are both locally coupled to the GFF Γ, and if they are conditionally independent given Γ, then $A \cup A'$ is also locally coupled to Γ.

   This is an immediate consequence of the definition and the previous properties of dyadic local sets: We note that $(A \cup A)_n = A_n \cup A'_n$, and that A_n and A'_n are 2^{-n}-dyadic local sets that are conditionally independent given Γ, so that $A_n \cup A'_n$ is also a 2^{-n}-dyadic local set (see item on the previous page).

2. Suppose now that A is a local set. Let us notice that if one knows A, then one knows all A_n's, and conversely, if one knows all A_n's for all $n \geq n_0$, then one knows also $A = \cap_{n \geq n_0} A_n$. Hence, the σ-field generated by A and the σ-field generated by $(A_n, n \geq n_0)$ coincide for all n_0.

   We now define for each n, the σ-field

   $$\mathcal{G}_n = \sigma(A, \Gamma_{A_n}) = \sigma(A_n, A_{n+1}, A_{n+2}, \ldots, \Gamma_{A_n}).$$

   Note that if one knows that $A_n = a$, $A_{n+1} = a'$ and Γ_a, then one knows also $\Gamma_{a'}$; hence,

   $$\mathcal{G}_n = \sigma(A_n, A_{n+1}, A_{m+2}, \ldots, \Gamma_{A_n}, \Gamma_{A_{n+1}}, \ldots)$$

   and $\mathcal{G}_{n+1} \subset \mathcal{G}_n$. In fact, the decomposition $\Gamma_a = \Gamma_{a'} + (\Gamma^{a'})_a$ shows that (for all μ)

   $$\Gamma_{A_{n+1}}(\mu) = E(\Gamma_{A_n}(\mu) | \mathcal{G}_{n+1}).$$

   In other words, for any $n \geq 1$,

   $$\Gamma_{A_n}(\mu) = E(\Gamma(\mu) | \mathcal{G}_n),$$

   so that $(\Gamma_{A_n}(\mu), n \geq 0)$ is an inverse martingale for the inverse filtration $\mathcal{G}_n$. Note that the random variable $\Gamma(\mu)$ is Gaussian, so that it is in any L^p for $1 \leq p < \infty$. It therefore converges almost surely and in any L^p to $E(\Gamma_{A_1}(\mu) | \mathcal{G}_\infty)$, where $\mathcal{G}_\infty := \cap_n \mathcal{G}_n$. We call this limit $\Gamma_A(\mu)$. We also define $\mathcal{F}_A = \mathcal{G}_\infty$.

   We then define $\Gamma^A(\mu)$ to be $\Gamma(\mu) - \Gamma_A(\mu)$, which is therefore also the limit of $\Gamma^{A_n}(\mu)$ (almost surely and in any L^p).

   There is now a little slaloming to be done to argue that there is a modification of the process so that when restricted on the complement of A, Γ_A is almost surely a harmonic function. Here is one way to proceed:

   For any $r < d(z, \partial O)$ and z, we first define for each n the circular average $\gamma_n(z,r)$ of Γ_{A_n} on the circle of radius r around z. Note that when the disc $D(z,r) \subset O_n$, then this circular average is equal to $h_{A_n}(z)$ because we know that h_{A_n} exists and is almost surely harmonic.

We have just seen that for each fixed z and r, $\gamma_n(z,r)$ converges almost surely to the circular average $\gamma_\infty(z,r)$ of Γ_A. Clearly, on the event where $d(z,\partial O) > 2r$, we have that almost surely $\gamma_\infty(z,r) = \lim_{n\to\infty} h_{A_n}(z)$ (which shows in particular that the sequence $h_{A_n}(z)$ converges almost surely on the event $\{d(z,\partial O) > 2r\}$). When $z \notin A$, we now define $h_A(z)$ to be the random variable $\lim_{n\to\infty} h_{A_n}(z)$.

Let us now argue that there exists a way to say that $z \mapsto h_A(z)$ can be modified into a continuous and harmonic on $D \setminus A$:

We know that for all z, z' that are at distance at least $2r$ from the boundary of D (using the L^6 convergence of $\gamma_n(z,r)$ to $\gamma_\infty(z,r)$ and then the conditional Jensen inequality), that for all r

$$\begin{aligned} E((\gamma_\infty(z,r) - \gamma_\infty(z',r))^6) &= \lim_{n\to\infty} E((\gamma_n(z,r) - \gamma_n(z',r))^6) \\ &= \lim_{n\to\infty} E\left(E(\gamma(z,r) - \gamma(z',r)|\mathcal{G}_n)^6\right) \\ &\leq E((\gamma(z,r) - \gamma(z',r))^6) = c_3 E((\gamma(z,r) - \gamma(z',r))^2)^3 \leq C(r)|z-z'|^3, \end{aligned}$$

from which we deduce that there exists (for each r) a continuous version of $z \mapsto \gamma_\infty(z,r)$. Since this is true for all rational r, and given the previous relation to $h_A(z)$, we deduce that there exists a version of $h_A(z)$ that is continuous on $D \setminus A$. Finally, because h_A is continuous, we can now easily deduce that it is also harmonic.

Hence, when A is a local set for the Gaussian Free Field Γ, then it comes automatically equipped with the two processes Γ_A and Γ^A, such that

- For all $\mu \in \mathcal{M}$, $\Gamma(\mu) = \Gamma_A(\mu) + \Gamma^A(\mu)$ almost surely.
- There exists a random harmonic function h_A in $D \setminus A$ such that for each measure $\nu \in \mathcal{M}$, $\Gamma_A(\nu) = \nu(h_A)$ almost surely on the event where the support of ν is in $D \setminus A$.
- The process Γ^A is the almost sure limit of the Γ^{A_n} as $n \to \infty$, where conditionally on the event $\{A_n = a\}$, Γ^{A_n} is a GFF in $D \setminus A_n$ that is independent of (A, Γ_A). Another way to phrase this it so say that the conditional distribution of Γ^{A_n} given (A, Γ^A) is just a function of A_n, and that it is a just a GFF in $D \setminus A_n$.

This last description of Γ^A is a way to give a precise meaning to the slightly loose statement that conditionally on A, Γ^A is a Gaussian Free Field in $D \setminus A$ that is independent of Γ_A (recall that limits in probability of Gaussian processes are necessarily Gaussian processes).

Let us give other ways to make this statement more precise. When O is an open set, one can define the law $\mathcal{L}_O$ of a GFF π in O. Then, indeed, we can say that the conditional law of Γ^A given $\sigma(A, \Gamma_A)$ is $\mathcal{L}_O$.

Note that all the information about the joint distribution of $(A, \Gamma_A, \Gamma^A, \Gamma)$ is encapsulated in the joint distribution of (A, Γ_A) (because we know the conditional distribution of Γ^A given (A, Γ_A) and $\Gamma = \Gamma_A + \Gamma^A$.

Conversely, suppose now that a random couple $(\tilde{A}, W)$ is defined on the same probability space, where:

- W is a random process indexed by $\mathcal{M}$, which is linear in w.r. to μ, such that with probability one, it can be be viewed as a harmonic function on the complement of A (in the previous

sense).

- If (on a possibly extended probability space), one defines a GFF π^A in the random set $D \setminus A$ that is conditionally independent of Γ_A given A, then $\Gamma = W + \pi^A$ is distributed like a GFF in D

Then, it is a simple exercise to check that A is a local set of Γ, and that $W = \Gamma_A$. The previous two properties therefore provide a characterization local sets.

Finally, let us point out the following immediate facts that shall be useful later on:

- Suppose that A is a local set that is coupled with a GFF Γ. If we observe A and Γ, then we know what the process Γ_A and therefore also what the harmonic function h_A is (just recall the description of Γ_A as limits of the Γ_{A_n}).
- If A and B are two local sets, and z some fixed point in O. Let $O_A(z)$ and $O_B(z)$ denote (if they exist) the connected components that contain z of $O \setminus A$ and $O \setminus B$. Then, almost surely on the event where $\{O_A(z) = O_B(z)\}$, one has $h_A(z) = h_B(z)$ (again just think of the definition of $h_A(z)$ as limits of Γ_{A_n}).

2.2.2 Comments on further classes of local sets

A further comment on unions of local sets. We have just pointed out that if A and B are two local sets that are conditionally independent, given the GFF that they are coupled with, then $A \cup B$ is also a local set of the GFF, but we did not describe the harmonic function $h_{A\cup B}$ in terms of h_A and h_B. There is one case where this description is easy to derive, namely when A and B are disjoint:

Lemma 2.1. *If A and B are two conditionally independent local sets, then for each $\varepsilon > 0$, almost surely on the event where $d(A, B) > 2\varepsilon$, $h_{A\cup B}$ is the unique harmonic function h on $D' = D\setminus(A\cup B)$ such that:*

- *$h - h_A$ goes uniformly to 0 when $z \to \partial D'$ with $d(z, B) > \varepsilon/2$.*
- *$h - h_B$ goes uniformly to 0 when $z \to \partial D'$ with $d(z, A) > \varepsilon/2$.*

Note that the fact that there exists at most one such function h follows immediately from the fact if $\tilde{h}$ was another such function, then $h - \tilde{h}$ would go uniformly to 0 on $\partial D'$ (on the event $d(A, B) > \varepsilon$). We already know that $h_{A\cup B}$ is harmonic, so (in order to prove the lemma) it suffices to prove that it does satisfy these two conditions. By symmetry of the roles played by A and B, we can restrict ourselves to the first one.

Let us now consider a deterministic set L consisting of the finite union of dyadic segments. Clearly $A \cup B \cup L$ and $A \cup L$ are both local sets. Furthermore, we know that on the event where A and B are separated by L in D, then the harmonic functions $h_{A\cup B\cup L}$ and $h_{A\cup L}$ coincide on the connected components that have part of A on their boundary (they are the limits of the same harmonic functions obtained from dyadic approximations).

On the other hand, because L is deterministic, it turns out that one can obtain $h_{A\cup L}$ in two steps: First observe h_A, then observe and add $(\Gamma^A)_L$ (note that because L is deterministic, there is no problem in defining this process). In this way, it is clear that on the event where the distance

between L and A is at least ε, then $h_A - h_{A\cup L}$ tend to 0 uniformly on the part of the boundary $D \setminus A$ that is at distance greater than $\varepsilon/2$ from L.

But, we can apply the same reasoning to $A \cup B$ instead of A, so that the same feature is true for $h_{A\cup B} - h_{A\cup B\cup L}$: It goes to 0 uniformly on the part of the boundary of $D \setminus (A \cup B)$ that is at distance greater than $\varepsilon/2$ from L.

Finally, we can observe that on the event where $d(A, B) > 2\varepsilon$, there exists almost surely a finite union L of dyadic segments with the property that it separates A from B in D, and is at distance greater than ε from A and remains always at distance smaller than $\varepsilon/2$ from B.

Putting the pieces together, we get that almost surely on the event where $d(A, B) > 2\varepsilon$, $h_{A\cup B} - h_A$ goes to zero uniformly when z tends to the part of the boundary of $D \setminus A$ that is at distance greater than ε from B.

Thin local sets. Let us focus on a particular class of local sets that will be useful later on: Let λ denote the two-dimensional Lebesgue measure, and λ_U denote the Lebesgue measure on an open set U in the plane. We say that the local set A is "thin" if for all open set U, one has almost surely

$$\Gamma_A(\lambda_U) = \int_{U\setminus A} h_A(x)\lambda(dx).$$

This implies that Γ_A is in fact fully characterized by h_A. Indeed, we know that it is possible to approximate any μ by a sequence μ_n, where each μ_n is a finite linear combination of Lebesgue measures of open sets, in such a way that $\Gamma(\mu)$ is the L^2 limit of the sequence $\Gamma(\mu_n)$. This implies that $\Gamma_A(\mu) = E(\Gamma(\mu)|\mathcal{F}_A)$ is the L^2 limit of the $\Gamma_A(\mu_n)$, which are in turn characterized by h_A.

An example of a thin local set is for instance a deterministic compact set of zero Lebesgue measure.

One may wonder whether a local set is thin as soon as it has zero Lebesgue measure (note that this is anyway a necessary condition). As we shall argue later in the lectures, this turns out not to be the case: There exists local sets A such that the Lebesgue measure of A is almost surely 0 and such that A is not thin. The goal of the next few paragraphs is to describe a simple criteria that ensures that a local set is thin.

It is easy to see that in order to see that A is thin, it suffices to check the previous identity for any open finite union U of dyadic squares in D.

Here is a simple criteria for a local set to be thin:

Proposition 2.2. *Suppose that A is a local set of the GFF Γ. Define $|A_n|$ to be the Euclidean area of A_n (which is 4^{-n} times the number of 2^{-n}-dyadic squares that A intersects). If almost surely, $|A_n| = o(1/n)$, then the local set A in thin.*

By the previous observation, we just need to check that under the previous conditions, if ν_U denotes the Lebesgue measure in a given finite union U of dyadic squares of D,

$$\Gamma_A(\nu_U) - \int_{U\setminus A} h_A(x)dx = 0.$$

It is easy to see that it amounts to check that $\Gamma_{A_n}(\nu_U 1_{A_n})$ tends to 0 as $n \to \infty$. Note that when n is sufficiently large, the set $A_n \cap U$ is just itself a finite union of 2^{-n}-dyadic squares, and the definition of Γ_{A_n} shows that $\Gamma_{A_n}(\nu_U 1_{A_n}) = \Gamma(\nu_U 1_{A_n}) = \Gamma(\nu_{U\cap A_n})$ (again, there is no difficulty in

defining these random variables because A_n can only take finitely values). We know that almost surely, $U \cap A_n$ consists of at most $N_n = 4^n |A_n| = o(4^n/n)$ dyadic squares.

The proposition will therefore be proven if we show that for some constant x_0, almost surely, for all large enough n, for all 2^{-n}-dyadic square U, $|\Gamma(U)| \leq x_0 n 4^{-n}$ – this will then imply that almost surely, $\Gamma_{A_n}(\nu_U 1_{A_n}) = o(1)$.

We can compute explicitly $E(\Gamma(\eta)^2)$ (via the double integral of $G_D(x,y)$) when η is the uniform distribution over any 2^{-n}-dyadic square om $[0,1]^2$, and see that there exists C such that for any n, and for any of the 2^{-n} dyadic squares,

$$E(\Gamma(\eta)^2) \leq C^2 n 4^{-2n}.$$

Hence, using the fact that $\Gamma(\eta)$ is a Gaussian random variable, we have the tail estimate

$$P(|\Gamma(\eta)| > CM\sqrt{n}4^{-n}) \leq \exp(-M^2/2).$$

Summing this over all 2^{-n}-dyadic squares, we see that for each n, the probability that there exists at least one 2^{-n}-dyadic square (out of the 4^n ones in $[0,1]^2$) for which $|\Gamma(\eta)| > CM\sqrt{n}4^{-n}$, is bounded by $4^n \exp(-M^2/2)$. If we choose $M = M(n) = x\sqrt{n}$, then for large enough (but given large x), this bound decays exponentially in n. Hence by Borel-Cantelli, we know that almost surely, for all large enough n, for all 2^{-n} dyadic square $|\Gamma(\eta)| \leq Cxn4^{-n}$, which concludes the proof.

In particular, if the Minkovski dimension of a local set A is strictly smaller than 2 (this is when $N_n = o((4-\varepsilon)^n)$, then it is a thin local set. A concrete simple effective criteria to check that a local set A is thin is therefore that

$$E(|A_n|) \leq ce^{-\varepsilon n}$$

for some fixed positive c and ε (which does indeed imply the almost sure condition $|A_n| = o(1/n)$ by Borel-Cantelli).

Actually, the criterion $E(|A_n|) = o(1/n)$ would be enough: Indeed, by Borel-Cantelli, one can find a deterministic sequence (n_k), such that almost surely, $|A_{n_k}| = o(1/n_k)$, from which one can then deduce (using the same proof as in the proposition, except that one just focuses on the subsequence n_k) that A is thin.

Let us make the following simple remark. Suppose that (A, h_A) defines a thin local set (i.e. adding to h_A a conditionally independent GFF in $D \setminus A$, one obtains a GFF Γ). Then $(A, -h_A)$ defines also a thin local set – indeed, the obtained field that one obtains by adding to $-h_A$ a conditionally independent GFF in $D \setminus A$ would then be distributed like $-\Gamma$.

In the sequel, we shall define some very particular thin local sets of the GFF (this will be for instance the SLE_4 curves): We say that the thin local set A is a *determining thin local set* when the harmonic function h_A is in fact some deterministic function of A. In these cases, one has just to successively sample two things in order to get the couple (A, Γ): First one samples A, and then, since h_A is automatically determined from A, one just needs to sample the GFF Γ^A in the complement of A. We can note that at this point, the existence of non-empty thin determining local sets is not obvious at all!

Monotone families of local sets. Let us now suppose that $(A^k, k \in \mathbb{Z})$ is a random non-decreasing (i.e. $A^k \subset A^{k+1}$) sequence of compact sets in $\overline{D}$, coupled to a Gaussian free field, such that for each k, A^k is a local set for Γ.

Let us define for all k, the random harmomic function h^k in $D \setminus A^k$ associated to A^k. We also define $A^\infty := \overline{\cup_k A^k}$ and $A^{-\infty} = \cap_k A^k$. Then:

Proposition 2.3. *The sets A^∞ and $A^{-\infty}$ are local sets for Γ.*

Proof. Let us first note that if for some n, all A^k's are almost surely 2^{-n}-dyadic local sets, then the result holds. Indeed, for all k and for all 2^{-n} deterministic dyadic local square a, we know that Γ^a is a GFF in $D \setminus a$, independent of $\sigma(A^k \subset a, \Gamma_a)$. Letting $k \to \infty$ or to $-\infty$, one gets immediately by monotone convergence that Γ^a is independent of $\sigma(A \subset a, \Gamma_a)$.

We now consider the general case: (A^k) is a non-decreasing family of local sets (not necessarily dyadic ones). Define, for each k and n, the set $A^k_n = (A^k)_n$. The sequence $(A^k_n, k \in \mathbb{Z})$ (for fixed n) is a non-decreasing family of dyadic 2^{-n} local sets, and we have just argued that $A^+_n := \cup_k A^k_n$ and $A^-_n := \cap_k A^k_n$ are local dyadic sets.

As $A^+_n = (A^{-\infty})_n$, it follows that $A^{-\infty}$ is local. We note also that $A \subset A^+_n \subset A_n$, and that it can happen that A^+_n is strictly smaller than A_n. However, each A^+_n is a dyadic local set, $A = \cap A^+_n$, and A^+_n is a deterministic function of A^+_{n+1}, which shows that $\Gamma_{A_n}(\mu)$ are inverse martingales with respect to the inverse filtration $\sigma(A_n, \Gamma_{A_n})$. These martingales converge almost surely and in any L^p as before, and we can then deduce that A^∞ is a local set.

□

Bibliographical comments

The notion of local sets of the GFF has been introduced by Schramm and Sheffield, who also stated most of the properties that we discussed here. The motivation for their work was precisely the relation with SLE that we will discuss in the next chapter. Our presentation differs a little bit from theirs, but the results and the main ideas are of course essentially equivalent.

Chapter 3

SLE_4, CLE_4 and the Gaussian Free Field

3.1 SLE_4 and the GFF

3.1.1 The characterizing property of SLE_4

In order to motivate the following couple of subsections and the introduction of SLE_4, let us briefly describe the main result that we will strive for, and its main consequence.

Suppose that $\gamma : [0,1] \to \overline{\mathbb{U}}$ is a random simple curve from -1 to 1 in the unit disk with $\gamma(0) = -1$, $\gamma(1) = 1$ and $\gamma(0,1) \subset \mathbb{U}$. For each $u \leq 1$, we define $\mathcal{F}_u$ the σ-field generated by $\gamma[0,u]$ and we let $U_u = \mathbb{U} \setminus \gamma[0,u]$. One can divide the boundary of U_u into two parts: The portion ∂_u^+ that goes clockwise from $\gamma(u)$ to -1 (this consists therefore of the "upper boundary of η and of the clockwise part of the unit circle between -1 and 1, and the anticlockwise portion ∂_u^-.

For each time t and $z \in U_u$, we can then define the harmonic measure $h_u^+(z)$ of ∂_u^+ in U_u at z: This is the probability that a planar Brownian motion started from z exits U_u through ∂_u^+.

Then, we shall now see that:

- There exists such a random curve, such that for each $z \in \mathbb{U}$, the process $(h_t^+(z), t \in [0,1])$ is a martingale with respect to the filtration $(\mathcal{F}_t)$. This will be the SLE_4 curve from -1 to 1 in $\mathbb{U}$.
- The SLE_4 curve from -1 to 1 in $\mathbb{U}$ is the only random curve (or more precisely, its law is) up to time-change the unique law on random curves with this property.

This characterizing property of SLE_4 will then enable us to view it as an interesting determining thin local set of the GFF (the harmonic function associated to $\gamma[0,u]$ will then be a constant times $h_u(z) - h_0(z)$).

3.1.2 Loewner chains background

We quickly review without proofs some basic facts about deterministic Loewner chains and simple curves in the upper half-plane.

- Suppose that $(\gamma(u), u \in [0, U))$ is a continuous (deterministic) simple curve in the closed upper half-plane $\overline{\mathbb{H}} = \{x + iy\ ,\ y \geq 0\}$ such that $\gamma(0) = 0$ and $\gamma(0, U) \subset \mathbb{H}$.

  Then, for each $u < U$, by Riemann's mapping theorem, one can define the two conformal transformations $\tilde{g}_u$ and $\tilde{f}_u$ from $\mathbb{H} \setminus \gamma(0, u]$ into $\mathbb{H}$, that are chosen so that

$$\tilde{f}_u(\gamma(u)) = 0,\ \tilde{f}_u(z) \sim z,\ \tilde{g}_u(z) = z + o(1) \text{ as } z \to \infty.$$

  Both these functions have a Laurent series expansion near ∞ with real-valued coefficients. For instance

$$\tilde{f}_u(z) = z + \tilde{W}_u + \tilde{a}(u)/z + o(1/z)$$

  (and one then has $\tilde{g}_u(z) = \tilde{f}_u(z) + \tilde{W}_u$ and $\tilde{g}_u(\gamma(u)) = \tilde{W}_u$) – note that this defines the function $\tilde{a}$ and $\tilde{W}$ from γ.

  It is easy to see that the mapping $u \mapsto 2\tilde{a}(u)$ is an increasing continuous function (that converges to some $\sigma \in (0, \infty]$ as $u \to U-$). We can therefore define the reparametrized continuous curve $\eta : [0, \sigma) \to \overline{\mathbb{H}}$ such that for all $u < U$, $\eta(2\tilde{a}(u)) = \gamma(u)$. We define now $f_t = \tilde{f}_{2\tilde{a}(u)}$, $g_t = \tilde{g}_{2\tilde{a}(u)}$ and $W_{2\tilde{a}(u)} = \tilde{W}_u$, so that for all .

  In summary, modulo a deterministic simple time-change, we can obtain this reparametrization η of the path for which $a(u) = 2t$, and then, for all $t \in [0, \sigma)$, $f_t(z) + W_t = g_t(z) = z + 2t/z + o(1/z)$. Note that if $|Im(\eta(t))|$ is unbounded, then one necessarily has $\sigma = \infty$.

- Loewner's equation provides a recepy to recover η out of the function $t \mapsto W_t = g_t(\eta(t))$ (which in particular shows that the curve η is in fact fully characterized by W). Indeed, for all $t \geq 0$, when $z \in \mathbb{H} \setminus \eta(0, t]$, it turns out that

$$\partial_t g_t(z) = 2/(g_t(z) - W_t).$$

  In particular, this enables (via the "reverse flow"), for each $y \in \mathbb{H}$ and each $T \geq 0$, to construct $g_T^{-1}(y)$ as the value at time T of the function $y(\cdot)$ with $y(0) = y$ and $\partial_t y(t) = -2/(y(t) - W(T - t))$. Then, one recovers $\eta(0, T]$ as $\mathbb{H} \setminus g_T^{-1}(\mathbb{H})$.

  Let us emphasize that for each simple curve η, there exists a continuous function W from which one can recover η uniquely using this procedure, but that if we are given any continuous W, it may happen that it does not correspond to a continuous curve η.

- Let us summarize a few trivial properties of the Loewner flow. If we fix $z \in \mathbb{H}$ and define $Z_t = X_t + iY_t := f_t(z)$ and $\theta_t := \arg(f_t(z)) \in (0, \pi)$ (as long as $z \notin \eta[0, t]$). Then,

$$Y_t - Y_0 = \int_0^t \Im(2/Z_s)ds, \quad X_t = Y_t/\arctan(\theta_t), \quad W_t = X_t - X_0 - \int_0^t \Re(2/X_s)ds.$$

  Hence, we see that W can obtained be a simple transformation starting from θ involving only addition or compositions with smooth functions. In particular, if we happen to know that (θ_t) is a semi-martingale with respect to some filtration (and that η is also adapted to this filtration – and therefore X and Y too), then it follows immediately that (W_t) is also a semi-martingale with respect to the same filtration.

- At each time t, the domain $H_t := \mathbb{H} \setminus \eta(0, t]$ is simply connected. Clearly, H_t is decreasing with t, so that the functions $t \mapsto G_{H_t}(x, y)$ are non-increasing. Recall that the Green's

function in $\mathbb{H}$ is given by

$$G_{\mathbb{H}}(x,y) = \frac{1}{2\pi}\log\frac{|x-\bar{y}|}{|x-y|} = \frac{1}{2\pi}\Re(\log(x-\bar{y}) - \log(x-y)).$$

By conformal invariance, we get

$$G_{\mathbb{H}\setminus\eta(0,t]}(x,y) = G_{\mathbb{H}}(f_t(x), f_t(y)) = G_{\mathbb{H}}(g_t(x), g_t(y)).$$

The functions $t \mapsto g_t(x)$ are smooth up to the possible finite time at which η hits x. Differentiating the previous expression with respect to t shows immediately that (for $x, y \in H_t$),

$$\partial_t G_{\mathbb{H}\setminus\eta(0,t]}(x,y) = -\frac{1}{2\pi} I_t(x) I_t(y),$$

where here and in the sequel, $I_t(x) = \Im(-2/f_t(x))$.

This shows in particular that for all smooth test function φ (continuous with compact support),

$$\int\int \varphi(x)\varphi(y)(G_{\mathbb{H}}(x,y) - G_{H_\infty}(x,y))d^2x d^2y = \int_0^\infty \int\int I_t(x) I_t(y)\varphi(x)\varphi(y) dt d^2x d^2y$$

(and in particular that the right-hand sided is bounded by $\int\int \varphi(x)\varphi(y)G_{\mathbb{H}}(x,y)d^2x d^2y$ uniformly for all η).

3.1.3 SLE$_4$ and the harmonic measure martingales

We are not going to give here a detailed construction of SLE$_4$, but just give a summary of some of its features. Let us state without proof the following result about SLE, that is essentially due to Rohde and Schramm:

Proposition 3.1. *For all $\kappa \leq 4$, there exists a random continuous simple curve η such that its corresponding driving function (W_t) is a one-dimensional Brownian motion running at speed $\sqrt{\kappa}$ (in other words, $\beta_t := W_t/\sqrt{\kappa}$ is a Brownian motion). Furthermore, almost surely, $\Im(\eta(t))$ is unbounded, $|\eta(t)| \to \infty$ as $t \to \infty$, and the Hausdorff dimension of η is equal to $1 + \kappa/8$ (and for each fixed $z \in \mathbb{H}$, $z \notin \eta[0,\infty)$ almost surely).*

Note that these are non-trivial statements. We will not need the full result about the Hausdorff dimension – in fact we will just use this proposition in the special case $\kappa = 4$ (and the fact that the Hausdorff dimension of η is strictly smaller than 2).

For an SLE$_4$ and each $z \in \mathbb{H}$, one therefore has $f_0(z) = z$ and

$$df_t(z) = -\sqrt{\kappa}d\beta_t + \frac{2}{f_t(z)}dt$$

as long as $f_t(z)$ does not hit 0. The only point z for which $f_t(z)$ hits 0 at time t is $z = \eta(t)$.

For each $z \in H_t$, we now define $\theta_t(z) \in (0,\pi)$ to be the argument of $f_t(z)$. This is clearly a continuous function of t (as long as $f_t(z)$ stays away from 0). Using Itô's formula, we get that

$$d\theta_t(z) = \Im(d\log(f_t(z))) = \Im(-\sqrt{\kappa}/f_t(z))d\beta_t + (2-\kappa/2)\Im(1/f_t(z)^2)dt.$$

This is why $\kappa = 4$ is very special. Let us highlight this very simple fact as a Proposition in order to highlight its importance:

Proposition 3.2. *When η is a SLE$_4$, then for all fixed $z \in \mathbb{H}$, $(\theta_t(z), t \geq 0)$ is a continuous martingale with respect to the filtration $(\sigma(\eta[0,t]))$.*

Note that for every given z, η almost surely does not hit z, so that the martingale $\theta_t(z)$ is defined for all times, and stays in $(0,\pi)$ (this explains why it is actually a martingale, and not only a local martingale). Of course, there exist random exceptional points that end up being on the curve, and for which the corresponding θ_t is not defined up to all positive times.

Let us now explain that the fact that $t \mapsto \theta_t$ are martingales does characterize the law of SLE$_4$. This will turn out to be useful later on. Suppose that $(\gamma_u, u < U)$ is a random simple curve in $\overline{\mathbb{H}}$ (with $\gamma(0) = 0$ and $\gamma(0,U) \subset \mathbb{H}$)) that is adapted with respect to some filtration $(\mathcal{F}_u)$, and such that U is a possibly infinite $(\mathcal{F}_u)$-stopping time. Define for each $z \in \mathbb{H}$, the conformal transformation $\tilde{f}_u$ from $\mathbb{H} \setminus \gamma(0,u)$ onto $\mathbb{H}$ with $\tilde{f}_u(\gamma_u) = 0$ and $\tilde{f}_u(z) \sim z$ as $z \to \infty$. Define $\varphi_u(z)$ to be the argument of $\tilde{f}_u(z)$. We will also assume that when $u \to U$, either $d(\gamma_u, \mathbb{R}) \to 0$ or $\Im(\tilde{g}_u(z))$ is unbounded.

Lemma 3.3. *Under these conditions, if for all $\varphi_u(z)$ is a martingale in the filtration $(\mathcal{F}_u)$, then γ is distributed like a time-changed SLE$_4$ from 0 to infinity in $\mathbb{H}$. It follows in particular that as $u \to U-$, $|\gamma(u)| \to \infty$.*

The proof of this lemma is simple; this is the same standard argument that is used in several cases in order to identify the scaling limit of some lattice model interface in terms of an SLE curve, using the fact that certain functions are local martingales with respect to this interface:

- First, time-change γ into η as before and for all $t \geq 0$, $\mathcal{G}_t = \mathcal{F}_{u(t)}$ where $u(t)$ is the inverse of $u \mapsto 2\tilde{a}(u)$. Since $u(t)$ is a stopping time for $(\mathcal{F}_u)$, it follows that for each fixed z, $\theta_t := \varphi(u(t))$ is a martingale (stopped at the stopping time $\sigma(U)$) for the filtration $(\mathcal{G}_t)$.
- Then, the previous considerations imply that W_t (stopped at σ) is a semi-martingale with respect to the filtration $(\mathcal{G}_t)$. This semi-martingale can be decomposed into its local martingale term M_t and its finite variation term V_t. Using the fact that $\theta_t = \Im(\log f_t(z))$, we can apply Itô's formula for semi-martingales, from which it follows that

$$d\theta_t = \Im(-dM_t/f_t(z)) + \Im(-1/f_t(z))dV_t + (2dt - d\langle M\rangle_t/2)\Im(1/f_t(z)^2)dt.$$

  As the finite variation part of this semi-martingale is zero, it follows that for each fixed z, for all $t \geq 0$,

$$\int_0^t \Im(1/f_s(z))dV_s + \Im(1/f_s^2(z))(2ds - d\langle M\rangle_s/2) = 0$$

  almost surely. By looking at $1/n$ times this quantity for $z = in$ and letting $n \to \infty$ (using the fact that almost surely, $f_s(in)/in$ converges uniformly to 1 in the time-interval $[0,t]$ as $n \to \infty$), we see that $V_t = 0$. By continuity of V with respect to t, we get that $V = 0$ at all times. Hence, we now know that for all given z and all $t \geq 0$,

$$\int_0^t \Im(1/f_s^2(z))(2ds - d\langle M\rangle_s/2) = 0$$

  Almost surely, for each t, one can find an n large enough such that $\Im(1/f_s^2(n+i))$ remains positive on $s \in [0,t]$, and one concludes that $\langle M\rangle_s - 4s = 0$ almost surely on $[0,t]$. By continuity with respect to t, we therefore conclude that $W_t/2$ behaves like a Brownian motion up to the time σ.

- Hence, it follows that the path η is a SLE_4 up to the time σ. But our conditions on the behavior of γ near U then imply that $\sigma = \infty$.

To conclude this subsection, let us reformulate and summarize its results in other simply connected domains than in the half plane. Suppose now that D is a simply connected domain in the plane, whose boundary is a curve (not necessarily simple, for instance D could be $\mathbb{H} \setminus [0, i]$), and consider two boundary points A and B of D (when the boundary is not a simple curve, the right notion of boundary "points" is in fact that of "prime ends" – for instance in the above example, $0+$ and $0-$ would correspond to different prime ends).

By Riemann's mapping theorem, there exists a conformal transformation Φ from $\mathbb{H}$ onto D, such that $\Phi(0) = A$ and $\Phi(\infty) = B$ (this is again defined in terms of prime ends). This conformal map is unique up to dilations of the upper half-plane (i.e. the only other such conformal maps are $z \mapsto \Phi(\lambda z)$ for $\lambda > 0$). We can now define the SLE_4 from A to B in D to be the image under Φ of an SLE_4 in the upper half-plane, as defined above. The fact that the law of SLE_4 in $\mathbb{H}$ is invariant under dilations (modulo reparametrization) shows that the law of SLE_4 from A to B is therefore well-defined, up to a multiplicative time reparametrization.

Suppose now that we have a random simple continuous curve $\gamma : [0, U) \mapsto \overline{D}$, such that $\gamma(0) = A$, $\gamma(0, T) \subset D$. For all $u < U$, the domain $D_u = D \setminus \gamma(0, u]$ is simply connected, and γ_u and B are boundary "points" of D_u. We can therefore divide its boundary in two portions from γ_u to B – we denote by ∂_u its clockwise portion. Define, for all $u \leq U$ and all $z \in D \setminus \gamma(0, u)$, the quantities $\mathrm{harm}_u(z)$ to be the harmonic measure of ∂_u in D_u at z (this harmonic measure is the probability that a Brownian motion started from z exits D_u through ∂_u).

In the particular case where $D = \mathbb{H}$, $A = 0$ and $B = \infty$, then $\mathrm{harm}_u(z) = \varphi_u(z)/\pi$.

Then, with all these notations:

1. If γ is defined to be an SLE_4 from A to B in D, then for each given $z \in \mathbb{H}$, $(\mathrm{harm}_u(z), u \geq 0)$ is a martingale (in the filtration $\sigma(\gamma[0, u])$).
2. If $(\gamma_u, u < U)$ is a random curve such that for each z, $\mathrm{harm}_u(z)$ is a (stopped) martingale with respect to some filtration, then γ is distributed like the time-change of a (stopped) SLE_4 from A to B in D.

3.1.4 The coupling with the GFF

From now on, we will assume that $\kappa = 4$ and that η is an SLE_4. Then, for each given z, $t \mapsto \theta_t(z)$ is a martingale, and one can note that as $t \to \infty$, it converges to $\theta_\infty(z)$ which is either 0 or π depending on whether η passes to the left of z or to the right of z. We see for instance that (viewing θ_t as a time-changed Brownian motion) that the quantity $\int_0^\infty I_t(z)^2 dt$ can be interpreted as the time at which a Brownian motion starting from $\theta_0(z)$ exits the interval $(0, \pi)$.

Recall the following classical fact: Let β is a standard Brownian motion and $H = (H_t)$ a continuous process adapted to the filtration of β, then the process $M_t = \int_0^t H_s d\beta_s$ is a local martingale that can be interpreted as a time-changed Brownian motion. Indeed, if one defines for all t, $U_t = \int_0^t H_s^2 ds$ and τ the inverse of U, and $\mathcal{G}_u = \mathcal{F}_{\tau(u)}$, then $B_u := M_{\tau(u)}$ is a Brownian motion with respect to the filtration $\mathcal{G}_u$ (possibly stopped at the stopping time U_∞ if this quantity is finite).

We will use this result in the particular case where there exists a deterministic U_0 such that almost surely, $U_\infty < U_0$. If we condition on B up to the stopping time U_∞ and then adds to $B_{U_\infty} = M_\infty$ a (conditionally) independent random Gaussian variable with mean 0 and variance $V := U_0 - U(\infty)$, one obtains (via the strong Markov property) a random variable that is distributed like a Brownian motion at time U_0 i.e. a Gaussian random variable with mean 0 and variance U_0.

Our goal is to construct a coupling of η with a GFF, so that η will be a local set of the GFF. As the Hausdorff dimension of η is almost surely strictly smaller than 2, η will necessarily be a thin local set, so that the law of the coupling will be totally encapsulated by the description of the harmonic function defined in $\mathbb{H} \setminus \eta$. The appropriate choice of the harmonic function turns out to be

$$h(z) = \frac{1}{\sqrt{2\pi}}(\theta_\infty(z) - \theta_0(z)).$$

Note that this is a very special case, because the harmonic function turns out to be deterministically determined from η (it is in fact not at all clear whether it is possible that such local sets exists).

Proposition 3.4. *This choice of η (an SLE_4) and h defines a thin local set of a GFF.*

Note that $\mathbb{H} \setminus \eta$ consists of two open simply connected domains H_- and H_+ that a respectively to the left and to the right of η. The proposition means that if one first samples η, and then (conditionally on η) samples two independent GFFs in the two domains H_- and H_+, and adds the harmonic function h, one has sampled exactly a GFF Γ in $\mathbb{H}$.

An obviously equivalent formulation is that if one adds $\sqrt{\pi/2} \times 1_{H_-}$ to $\tilde{\Gamma}$, one gets $\Gamma + l(\cdot)$, where l is the harmonic function in $\mathbb{H}$ with boundary values $\sqrt{\pi/2} \times 1_{\mathbb{R}_-}$ on the real line.

In order to prove this statement, it suffices to prove that for any fixed continuous function with compact support φ, the random variable $\tilde{\Gamma}(\varphi) + \int \varphi(z)h(z)d^2z$ is a Gaussian random variable with mean 0 and variance

$$U_0 := \int\int G_{\mathbb{H}}(x,y)\varphi(x)\varphi(y)d^2xd^2y$$

(indeed, this will describe for all finite family $\varphi_1, \ldots, \varphi_k$ the characteristic function of $(\Gamma(\varphi_1), \ldots, \Gamma(\varphi_k))$).

Recall that conditionally on η, the conditional distribution of $\tilde{\Gamma}(\varphi)$ is a centered normal with variance

$$V := \int\int G_{\mathbb{H}\setminus\eta}(x,y)\varphi(x)\varphi(y)d^2xd^2y.$$

Let us now define

$$M_t := \frac{1}{\sqrt{2\pi}} \int \varphi(z)\theta_t(z)d^2z.$$

By (stochastic) Fubini, we get that

$$M_t = \int_0^t H_s d\beta_s, \text{ where } H_s := \frac{1}{\sqrt{2\pi}} \int \varphi(z)I_s(z)d^2z.$$

Hence, one can view M_∞ as a Brownian motion B, stopped at the time $U_\infty := \int_0^\infty H_s^2 ds$.

In other words, integrating this with respect to $\varphi(x)d^2x$, $\varphi(y)d^2y$ and from time 0 to infinity, we get

$$U_0 - V = \int\int (G_{\mathbb{H}}(x,y) - G_{\mathbb{H}\setminus\eta}(x,y))\varphi(x)\varphi(y)d^2xd^2y = \int_0^\infty H_s^2 ds = U_\infty,$$

which concludes the proof.

Proposition 3.5. *Suppose that one couples an SLE$_4$ with the GFF Γ using the previously described coupling. Then, for any fixed t, the curve $\eta[0,t]$ is also a thin local set w.r. to the same GFF Γ. The harmonic function associated to $\eta[0,t]$ is*

$$h_t(z) = \frac{1}{\sqrt{2\pi}}(\theta_t(z) - \theta_0(z)).$$

This result shows that η is in fact a "continuously increasing family of local sets". Heuristically, these are continuous counterpart of the ways to describe iteratively discrete local sets, by uncovering the value of the GFF at different sites one by one.

In order to prove this proposition, we need to recall the following property, that follows immediately from the Markov property of Brownian paths: Suppose that $t > 0$ is fixed, and let us sample $\eta[0,t]$. Then, the random path $\eta^t := (f_t(\eta(t+s)), s \geq 0)$ is an SLE$_4$ that is independent of $\eta[0,t]$.

In particular, we can apply the previous proposition to this random path: If we first condition on $\eta[0,t]$, then in the remaining domain (in order to construct Γ), one samples an SLE$_4$ from $\eta(t)$ to ∞, then two independent Gaussian Free Fields in each of the two connected components speraterd by η^t. The sum of these GFF and the harmonic function defined by η^t is then exactly a GFF in the complement of $\eta[0,t]$, and it is independent of $\eta[0,t]$. Hence, we see that $\eta[0,t]$ is coupled as a local set to the same GFF Γ.

Finally, let us notice that it is also possible by symmetry to define what we refer to as "other" coupling between SLE$_4$ and the GFF by simply using the harmonic function $-h$ instead of h (this works as the obtained field is the distributed like the negative of a GFF i.e. like a GFF).

3.1.5 SLE$_4$ is a deterministic function of the GFF

We now explain the following important property of the above-described coupling of SLE$_4$ with the GFF.

Proposition 3.6. *In the above-described coupling between a SLE$_4$ and the GFF, the SLE$_4$ is in fact a deterministic function of the field.*

In fact, a by-product of the proof will be the following non-trivial feature of SLE$_4$:

Proposition 3.7. *The law of η is reversible, in the following sense: Up to time-reparametrization, an SLE$_4$ from -1 to 1 in $\mathbb{U}$ is distributed like an SLE$_4$ $\tilde{\eta}$ from 1 to -1 in $\mathbb{U}$.*

Let us first summarize the outline of the proof: Define on the same probability space, a triple $(\eta, \hat{\eta}, \Gamma)$, such that η is an SLE$_4$ from -1 to 1, $\hat{\eta}$ is an SLE$_4$ from 1 to -1, Γ is a GFF in $\mathbb{U}$ such that

- The joint law of (η, Γ) is that of the first coupling between the SLE$_4$ and the GFF as described above.
- The joint law of $(\hat{\eta}, \Gamma)$ is that of the second coupling between the SLE$_4$ from -1 to 1 and the GFF as described above.
- Conditionally on Γ, the two paths η and $\hat{\eta}$ are independent.

The following lemma will imply the proposition:

Lemma 3.8. *In this construction, the trace of η is almost surely equal to the trace of $\hat{\eta}$.*

Indeed, if the lemma is true, then the trace of η is conditionally independent to itself, given Γ (because it is conditionally independent to the trace of $\hat{\eta}$). It is therefore a deterministic function of Γ. Furthermore, because η is the time-reversal of $\hat{\eta}$, we see that the time-reversal of an SLE_4 is indeed an SLE_4.

We know that almost surely, both η and $\hat{\eta}$ are simple curves. It will therefore be sufficient to prove that for all given $s \geq 0$, the point $\hat{\eta}(s)$ is almost surely on η (this implies that this holds simultaneously for a countable dense set of times s, and therefore for all times, by continuity). We will prove this by showing that conditionally on $\hat{\eta}(0, s]$, the path η up to the time t at which it hits $\hat{\eta}[0, s]$ is distributed like an SLE_4 from 0 to $\hat{\eta}(s)$ in $\mathbb{U} \setminus \hat{\eta}(s)$. Hence, η does indeed hit $\hat{\eta}(s)$.

In order to prove that conditionally on $\hat{\eta}[0, s]$, the path $\eta[0, T)$ is an SLE_4 from -1 to $\hat{\eta}(s)$, it suffices the check that the previously described characterization of SLE_4 (in terms of martingales) holds.

Let us now fix s, and define for all $t > 0$, the σ-field $\mathcal{F}_t = \sigma(\eta[0, t], \hat{\eta}[0, s])$. We know that $\eta[0, t] \cup \hat{\eta}[0, s]$ is a local set, as union of conditionally independent local sets. Let us denote by $h_{t,s}$ the corresponding harmonic function (so that $h_t := h_{t,0}$ is the harmonic function associated to $\eta[0, t]$ and $\hat{h}_s = h_{0,s}$ is the harmonic function associated to $\hat{\eta}[0, s]$).

But by Lemma 2.1, we know what $h_{t,s}$ is as long as $\eta[0, t] \cap \hat{\eta}[0, s]$: It is equal to the harmonic function with boundary values depicted on Figure XX, minus the harmonic function $h_{0,0}$. It therefore follows that conditionally on $\hat{\eta}[0, s]$, the harmonic function $h_{t,s}$ is exactly the one that caracterizes SLE_4 from -1 to $\hat{\eta}(s)$ in the complement of $\hat{\eta}[0, s]$, which concludes the proof.

3.2 Still to be written-up

CLE4 construction and coupling with the GFF, properties of this coupling, CLE4 and the square of the GFF, CLE4 and Liouville QG. Bibliographic references to Schramm-Sheffield, Sheffield, Miller-Sheffield, Dubédat etc.

Further discussion of the coupling of the GFF with other SLE_κ's. Girsanov and absolute continuity for the GFF, example of $\mathrm{SLE}_\kappa(\rho)$.