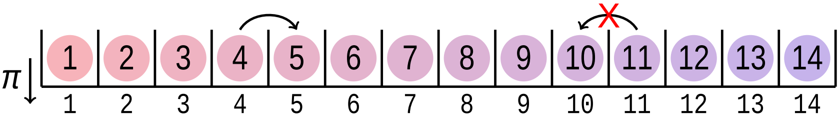
Coxeter Group Actions on Interacting Particle Systems, part II

Axel Saenz

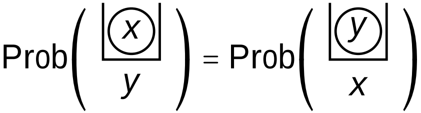
University of Warwick

Color position symmetry



Theorem (Thm. 1.4 [AAV11])

Take the colored ASEP with step initial conditions. Let $Y_n(t)$ be the color of the particle at position $n \in \mathbb{Z}$ at time $t \geq 0$, and let $X_n(t)$ be the position of the particle of color $n \in \mathbb{Z}$ at time $t \geq 0$. Then, for any fixed $t \geq 0$, the process $\{Y_n(t)\}_{n \in \mathbb{Z}}$ has the same distribution as $\{X_n(t)\}_{n \in \mathbb{Z}}$.



ASEP-$(q; \mathrm{N}; \mathrm{m})$: configuration

The *asymmetric simple exclusion process (ASEP)-*$(q; \mathrm{N}; \mathrm{m})$:

- N particles
- a lattice $\mathrm{L} = \{1, 2, \dots, M\}$
- exactly m_x particles occupy site $x \in \mathrm{L}$ with
- exactly N_i particles of color i particles in the system.

We write $\mathrm{m} = (m_x)_{x \in \mathrm{L}}$ and $\mathrm{N} = (N_i)_{i=1}^{N}$.

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$$k = (k_x^{(i)})_{x \in L; i \in [\![1, N]\!]}, \tag{1}$$

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Example: Take $N = 3$, $\mathbf{N} = (2,1)$, and $\mathbf{m} = (1,2)$. We have only two possible configurations.

We have two natural projections: the *color-blind* projection , and the *fusion* projection .

$$S(1,1) \to S(N,1) \to S(N,m). \tag{2}$$

Projections

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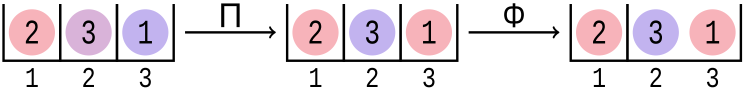
$$S(1;1) \quad ! \quad S(N;1) \quad ! \quad S(N;m): \qquad (2)$$

Example: Take $N = 3$, $N = (2;1)$, and $m = (1;2)$.



Pullback dynamics

We introduce a stochastic map $\quad : \mathrm{S}(\mathrm{N}, \mathrm{m}) \to \mathrm{S}(1, 1)$ (details later). Let L be the generator for the ASEP-$(q, 1, 1)$, and let $\hat{L}$ be the generator for the ASEP-$(q, \mathrm{N}, \mathrm{m})$. Then,

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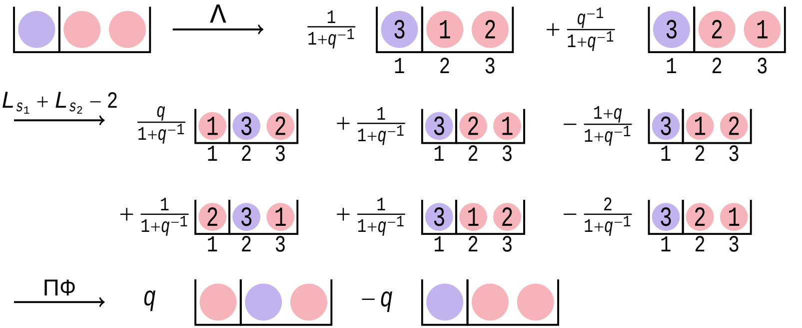
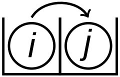


Set $[k]_q = 1 + \cdots + q^{k-1}$. The rate for a particle of color i at site $x \in L$ to swap position with a particle of color $i < j$ at site $x+1 \in L$ is given by

$$\frac{q^{\sum_{s>i} k_x^{(s)}}[k_x^{(i)}]_{q^{-1}}}{[m_x]_{q^{-1}}} \cdot \frac{q^{\sum_{r<j} k_{x+1}^{(r)}}[k_{x+1}^{(j)}]_{q^{-1}}}{[m_{x+1}]_{q^{-1}}}, \qquad (4)$$



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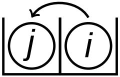

Coexeter group: representation

A *Coexeter group* (W, S) is a group W with a generating set $S = \{s_i \mid i \in I\}$ and relations

$$(s_i s_j)^{m_{i,j}} = 1 \tag{6}$$

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Example: The symmetric group S_N is the Coexeter A_{N-1} group with generating set $S = \{s_1, \ldots, s_{N-1}\}$ and

$$m_{i,j} = \begin{cases} 1, & i = j \\ 3, & |i - j| = 1 \\ 2, & |i - j| > 1 \end{cases}. \tag{8}$$

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In particular, we have $s_i = (i\ i+1)$ and

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The length function is equal to the number of inversions

$$\ell(w) = \#\{(i,j) \mid i < j, w(i) > w(j)\}. \tag{10}$$

Left cosets

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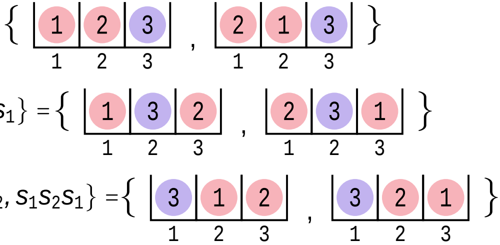

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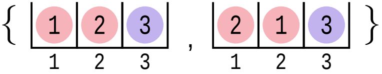
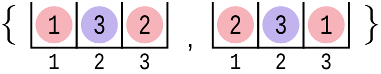
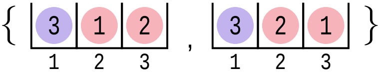
$H = \{e, s_1\} =$ 

$s_2 H = \{s_2, s_2 s_1\} =$ 

$s_1 s_2 H = \{s_1 s_2, s_1 s_2 s_1\} =$ 

For a fixed coset H, each configuration is the same in the state space $S(\mathrm{N}, 1)$ with $\mathrm{N} = (2, 1)$.

Take a pair of *(parabolic)* subgroups $H, H' < W$ of the Coexeter group (W, S). The set of minimal length representative for double cosets $H' \ H$ is given by

$$D_{H',H} := D_{H'}^{-1} \cap D_H. \tag{11}$$

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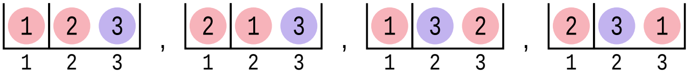
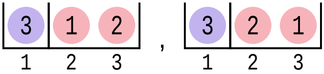
$$D_{H^{0};H} := D_{H^{0}}^{\;1} \setminus D_{H}: \tag{11}$$

Example:Take $N = 3$, $W = S_N$, $H = S(2) \quad S(1) = hs_1 i$, and $H^{0} = S(1) \quad S(2) = hs_2 i$. Then, $D_H = fe; s_1 s_2 g$:

$$H'eH = \{e, s_1, s_2, s_2 s_1\}$$

$= \{$  $\}$

$H' s_1 s_2 H = \{s_1 s_2, s_1 s_2 s_1\} = \{$  $\}$

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Example: Take $N = 3$, $W = S_N$, $H = S(2) \times S(1) = \langle s_1 \rangle$, and $H' = S(1) \times S(2) = \langle s_2 \rangle$. Then, $D_H = \{e, s_1 s_2\}$.

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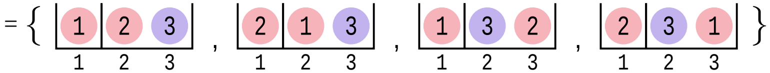

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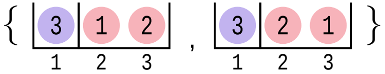

For a fixed double coset $H' \backslash W / H$, each configuration is the same in the state space $\mathrm{S}(\mathrm{N}, \mathrm{m})$ with $\mathrm{N} = (2, 1)$ and $\mathrm{m} = (1, 2)$.

Lemma

The stationary distribution for ASEP-$(1, 1)$ is

$$\pi(w) = \frac{q^{\ell(w)}}{\prod_{k=1}^{N-1}[k+1]_{q^{-1}}} \tag{12}$$

for $w \in S_N$.

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Proof: Take a configuration $w \in S_N$. For any $\tilde{w} \in S_N$ with $\ell(\tilde{w}) = \ell(w) + 1$, we have the following rates

$$\tilde{w} \xrightarrow{q} w \xrightarrow{1} \tilde{w}. \tag{13}$$

The resulting current is zero:

$$q\,\pi(\tilde{w}) - \pi(w) = q^{1-\ell(\tilde{w})} - q^{-\ell(w)} = 0. \tag{14}$$

Fact: ([Hum90, Car85]) For each $w \; 2 \; H^0 \quad H$ with $\quad 2 \; D_{H^0;H}$, there is a unique expression

$$w = a \quad b; \quad a \; 2 \; H^0 \setminus D_L; b \; 2 \; H \tag{15}$$

so that $L = \quad {}^1H$ and $`(w) = `(a) + `( \quad ) + `(b)$.

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so that $L = \quad {}^1H$ and $`(w) = `(a) + `(\quad) + `(b)$. Then,

$$\mathrm{Prob}\,(\quad(\quad) = a\quad b) = \frac{q^{\quad `(a)\quad `(b)}}{\sum_{a\quad b 2 H^0\quad H} q^{\quad `(a)\quad `(b)}}: \qquad (16)$$

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Lemma

The stationary distribution for ASEP-(N; m) is

$$\mathrm{e}(\;) = q^{\;\ell(\;)} \frac{\sum_{a \; b \in H^0 \; H} q^{\;\ell(a)\;\ell(b)}}{\prod_{k=1}^{N\;1} [k+1]_{q^{\;1}}} \qquad (17)$$

for $\; \in D_{H^0;H}$*. Moreover, we have*

$$\mathrm{e} \;\; = \qquad (18)$$

with $\;$ *the stationary distribution of ASEP-(1; 1).*

Color-position symmetry

We have the following bijections

$$
\begin{array}{ccccc}
\mathcal{S}(1,1) & \xrightarrow{\Pi} & \mathcal{S}(\mathsf{N},1) & \xrightarrow{\Phi} & \mathcal{S}(\mathsf{N},\mathsf{m}) \\
\uparrow \theta & & \uparrow \theta^{'} & & \uparrow \theta^{''} \\
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Theorem (Thm. 3.8 [Kua20])

Let k_t 2 S(N; m) *and* h_t 2 S(m; N) *be two independent multi-species ASEPs. Set* $H = S(N)$ *and* $H^0 = S(m)$ *with the bijections*

$$ {}^{00}_{1} : D_{H^0;H} \ ! \ \mathrm{S}(N;m); \qquad {}^{00}_{2} : D_{H;H^0} \ ! \ \mathrm{S}(m;N): \tag{19}$$

Then,

$$\mathrm{Prob}(k_t = {}^{00}_{1}(\quad)) = \mathrm{Prob}(h_t = {}^{00}_{2}(\quad {}^{1})) \tag{20}$$

for all 2 $D_{H^0;H}$.

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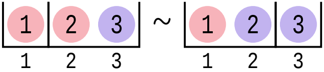

Proof

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For each $w \in H' \quad H$ with $\quad \in D_{H',H}$, there is a unique expression

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so that $L = \quad ^{1}H \quad$ and $\ell(w) = \ell(a) + \ell(\quad) + \ell(b)$. Recall the stochastic map $\quad : D_{H',H} \quad \to \quad W$ with

$$\text{Prob}\,(\quad(\quad) = a \quad b) \propto q^{\quad \ell(a) \quad \ell(b)}. \tag{22}$$

Take the generator $\hat{L}$ for the multi-species ASEP(q; N; m) and the generator L for the multi-species ASEP(q; 1; 1). Then,

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The result follows from the same coupling argument with the time-reversed process and the following identities

$$= \; ; \quad = \; ; \quad = \; ; \quad L_{s;x} = \hat{L}_{s;x} \tag{24}$$

so that is the inversion map.

Thank you for your attention!

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