Integrable systems and duality

Based on Chen, de Gier, Wheeler

"Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation"

Outline

- The notion of the duality: operator / matrix formalism.
- Multi-species ASEP process.
- Hecke algebra, tKZ equation and trivial duality function.
- Non-symmetric Macdonald polynomials.
- ASEP polynomials: construction of non-trivial duality functions.
- Matrix product Ansatz and examples.
- Global duality.

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Duality for Markov processes: Definition

- Given two Markov processes on state spaces A and B,

$$X = (\Omega_A, \mathcal{F}_A, (X_t)_{t\geqslant 0}, \{\mathbb{P}_a\}_{a\in A}), \qquad Y = (\Omega_B, \mathcal{F}_B, (Y_t)_{t\geqslant 0}, \{\mathbb{P}^b\}_{b\in B}).$$

- A function $\Psi : A \times B \to \mathbb{C}$ is a duality function for X and Y iff:
- Let (P_t) and (Q_t) be semi-groups of (X_t) and (Y_t). Recall: $P_t f(a) = \mathbb{E}_a[f(X_t)]$.
- Assume that the infinitesimal generators L and M are well-defined.

$$\begin{aligned} \forall a \in A, b \in B, \forall t \geqslant 0, \quad & \mathbb{E}_a[\Psi(X_t, b)] = \mathbb{E}^b[\Psi(a, Y_t)]. \\ \iff \quad & P_t\Psi(\cdot, b)(a) = Q_t\Psi(a, \cdot)(b). \\ \iff \quad & L[\Psi(\cdot, b)](a) = M[\Psi(a, \cdot)](b). \end{aligned}$$

- Write

$$\left.\begin{aligned} L[\Psi(\cdot, b)](a) &= \sum_{a'\in A} \ell(a, a')\Psi(a', b), \\ M[\Psi(a, \cdot)](b) &= \sum_{b'\in B} m(b, b')\Psi(a, b'). \end{aligned}\right\} \text{ left actions} \qquad \text{(Gen)}$$

Duality for Markov processes: Definition

$\Psi : A \times B \to \mathbb{C}$ is a duality function.

$\Updownarrow$

$$\forall a \in A, b \in B, \quad L[\Psi(\cdot, b)](a) = M[\Psi(a, \cdot)](b)$$

where

$$\begin{aligned} L[\Psi(\cdot, b)](a) &= \sum_{a' \in A} \ell(a, a')\Psi(a', b), \\ M[\Psi(a, \cdot)](b) &= \sum_{b' \in B} m(b, b')\Psi(a, b'). \end{aligned} \tag{Gen}$$

Duality for Markov processes: Matrix formalism

- Consider vector spaces with elements of A and B as bases,

$$\mathbb{A} = \text{Span}\{|a\rangle, a \in A\}, \qquad \mathbb{B} = \text{Span}\{|b\rangle, b \in B\}.$$

- Define an element in the tensor product $|\Psi\rangle \in \mathbb{A} \otimes \mathbb{B}$,

$$|\Psi\rangle = \sum_{\substack{a \in A \\ b \in B}} \Psi(a, b) |a\rangle \otimes |b\rangle.$$

- Define linear operators $\mathbb{L} \in \text{End}(\mathbb{A})$ and $\mathbb{M} \in \text{End}(\mathbb{A})$,

$$\mathbb{L}\,|a\rangle = \sum_{a' \in A} \ell(a', a)\,|a'\rangle, \qquad \mathbb{M}\,|b\rangle = \sum_{b' \in B} m(b', b)\,|b'\rangle,$$

where coefficients $\ell(a', a)$ and $m(a', a)$ are given by (Gen).

- $\mathbb{L}$ and $\mathbb{M}$ are right actions.

Duality for Markov processes: Matrix formalism

Prop. 1.1 : Duality relation is equivalent to $\mathbb{L}|\Psi\rangle = \mathbb{M}|\Psi\rangle$.

Proof: Direct computation.

$$\mathbb{L}|\Psi\rangle = \sum_{a'\in A} \sum_{\substack{a\in A\\ b\in B}} \ell(a', a)\Psi(a, b) \left|a'\right\rangle \otimes |b\rangle$$

$$= \sum_{\substack{a\in A\\ b\in B}} \underbrace{\sum_{a'\in A} \ell(a, a')\Psi(a', b)}_{L[\Psi(\cdot,b)](a)} |a\rangle \otimes |b\rangle \,.$$

$$\mathbb{M}|\Psi\rangle = \sum_{b'\in A} \sum_{\substack{a\in A\\ b\in B}} m(b', b)\Psi(a, b) |a\rangle \otimes \left|b'\right\rangle$$

$$= \sum_{\substack{a\in A\\ b\in B}} \underbrace{\sum_{b'\in B} m(b, b')\Psi(a, b')}_{M[\Psi(a,\cdot)](b)} |a\rangle \otimes |b\rangle \,.$$

□

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mASEP on the segment

The multi-species asymmetric simple exclusion process (mASEP):

- On the segment $\{1, \ldots, n\}$.
- At most one particle per site.
- Particles of type $1, \ldots, r$. Empty site is of type 0.
- Higher-type particle can switch with a lower-type particle.
- Switch rates are: τ_{right} to the right, τ_{left} to the left. For $i > j$,

- ASEP $\Leftrightarrow r = 1$.

mASEP on the segment

A configuration can be encoded in two ways.

Example for $n = 5$:

Monomial: $z^\nu \in \mathbb{C}[z_1, \ldots, z_n]$. $\rightsquigarrow$ Example: $z_2 z_3^2 z_5$.

- $A = \left\{ z^\nu := \prod_{i=1}^{n} z_i^{\nu_i}, 0 \leqslant \nu_i \leqslant r \right\}$.
- $\mathbb{A} = \{P \in \mathbb{C}[z_1, \ldots, z_n]$ s.t. the degree in each variable is at most $r\}$.

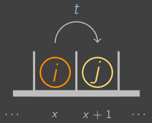
Sequence : $\mu = (\mu_i)_{1 \leqslant i \leqslant n} \in \{0, \ldots, r\}^n$. $\rightsquigarrow$ Example: $|0, 1, 2, 0, 1\rangle$.

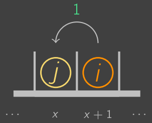
- $B = \left\{ |\mu_1 \ldots \mu_n\rangle := \bigotimes_{i=1}^{n} |\mu_i\rangle_i , 0 \leqslant \mu_i \leqslant r \right\}$
- $\mathbb{B} = \bigotimes_{i=1}^{n} V_i$ where $V_i \cong \mathbb{C}^{r+1}$ and $\{|0\rangle, \ldots, |r\rangle\}$ = canonical basis of $\mathbb{C}^{r+1}$.

Particle content: $0^{m_0} 1^{m_1} \ldots r^{m_r}$, $m_i = |\{k : \mu_k = i\}|$. $\rightsquigarrow$ Example: $0^2 1^2 2^1$.

mASEP on the segment: Local generator

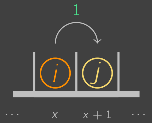
- Fix $t > 0$.
- Consider two processes:
  - $\tau_{\text{right}} = t$ and $\tau_{\text{left}} = 1$:

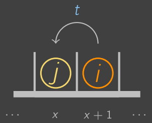




for $i > j$.

  - $\tau_{\text{right}} = 1$ and $\tau_{\text{left}} = t$:





for $i > j$.

- The forward dynamics of one is the backward dynamics of the other.
- Denote their generators by $\mathbb{L}$ and $\mathbb{M}$.
- Initial conditions might be different!

mASEP on the segment: Local generator

- The local generators $\mathbb{L}_i$ and $\mathbb{M}_i$ act on particles at positions i and $i+1$.

$$\mathbb{L} = \sum_{i=1}^{n-1} \mathbb{L}_i, \qquad \mathbb{M} = \sum_{i=1}^{n-1} \mathbb{M}_i.$$

- The local operators L_i and M_i are,

$$L_i[f](\nu) = \sum_{\nu'} \underbrace{\ell_i(\nu, \nu')}_{\tau(\nu \to \nu')} f(\nu'), \qquad M_i[f](\mu) = \sum_{\mu'} \underbrace{m_i(\mu, \mu')}_{\tau(\mu \to \mu')} f(\mu').$$

- Example for ASEP ($r = 1$):

$$L_i = (\ell_i(\nu, \nu'))_{\nu,\nu'} \qquad M_i = (m_i(\mu, \mu'))_{\mu,\mu'}$$

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & 0 & 0 \\ 01 & 0 & -1 & 1 & 0 \\ 10 & 0 & t & -t & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & 0 & 0 \\ 01 & 0 & -t & t & 0 \\ 10 & 0 & 1 & -1 & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array}$$

mASEP on the segment: $r = 1$, ASEP

- Define $\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)$, where s_i acts on polynomials by $z_i \leftrightarrow z_{i+1}$.

Prop. 2.1 : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i\,|\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu)\,|\nu'\rangle$.

Proof:

- $\mathbb{L}_i\,|1\rangle = \mathbb{L}_i\,|z_i z_{i+1}\rangle = 0$ due to the symmetry.
- $\mathbb{L}_i|z_i\rangle = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(z_{i+1} - z_i) = z_{i+1} - tz_i$.

- $\mathbb{L}_i|z_{i+1}\rangle = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(z_i - z_{i+1}) = tz_i - z_{i+1}$.

$L_i = (\ell_i(\nu, \nu'))_{\nu,\nu'}$

	00	01	10	11
00	0	0	0	0
01	0	−1	1	0
10	0	t	$-t$	0
11	0	0	0	0

□

mASEP on the segment: Local duality

- A function $\Psi : A \times B \to \mathbb{C}$ is a local duality function for X and Y iff:

$$\mathbb{L}_i|\Psi\rangle = \mathbb{M}_i|\Psi\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{LD}$$

  where

$$|\Psi\rangle = \sum_{\nu \in A} \sum_{\mu \in B} \Psi(\nu, \mu) \prod_{k=1}^{n} z_k^{\nu_k} |\mu\rangle.$$

- Local duality ⇒ global duality. (The reverse is false in general.)
- We focus on the local duality, i.e. solutions of (LD).

mASEP on the segment: Local duality

- The following duality functions for ASEP are constructed in [BCS+14] [1].

Theorem 4.1 & 4.2 [Duality functions of ASEP on $\mathbb{Z}$]:

- ASEP occupation process $(\nu_x)_{x\in\mathbb{Z}}$ with rates $(\tau_{\text{right}}, \tau_{\text{left}}) = (1, t)$;
- Position process $\vec{x}(\mu) = (x_1(\mu) < \cdots < x_n(\mu))$ with rates $(\tau_{\text{right}}, \tau_{\text{left}}) = (t, 1)$.

$$\Psi(\nu, \mu) = \prod_{x\in\vec{x}(\mu)} \left(\prod_{i<x} t^{\nu_i} \right) \nu_x ,$$

$$\Psi(\nu, \mu) = \prod_{x\in\vec{x}(\mu)} \left(\prod_{i\leqslant x} t^{\nu_i} \right) .$$

- The first one can be recovered using the framework of [2] but not the second one.

[1] From duality to determinants for q-TASEP and ASEP (2014)

[2] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

mASEP on the segment: Something wrong...

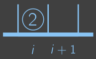
- Define $\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)$, where s_i acts on polynomials by $z_i \leftrightarrow z_{i+1}$.

Prop. 2.1 : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i\,|\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu)\,|\nu'\rangle$.

- $\mathbb{L}_i$ acts on polynomials faithfully for $r = 1$.
- For $r \geqslant 2$, consider the local dynamics starting from 

.

$$\mathbb{L}_i \left| z_i^2 \right\rangle = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)\cdot z_i^2 = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(z_{i+1}^2 - z_i^2)$$
$$= z_{i+1}^2 + (1-t)z_i z_{i+1} - t z_i^2.$$

- But the dynamics gives 

$\rightsquigarrow 1\cdot$ 

$- t\cdot$ 

.

mASEP on the segment: Something wrong...

- Define $\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)$, where s_i acts on polynomials by $z_i \leftrightarrow z_{i+1}$.

Prop. 2.1 : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i |\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu) |\nu'\rangle$.

- $\mathbb{L}_i$ acts on polynomials faithfully for $r = 1$.
- Conclusion: the basis $\left\{ |\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i} \right\}$ is not a good basis to work with.
- We look for a basis of polynomials $\{|\nu\rangle = f_\nu(z)\}$ such that

$$\mathbb{L}_i |\nu\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) \left|\nu'\right\rangle, \qquad \forall \nu. \qquad \text{(Adm)}$$

We say that $\{|\nu\rangle = f_\nu(z)\}$ is admissible.

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mASEP on the segment: Hecke algebra

- Type A_{n-1} Hecke algebra with generators $\{T_i\}_{1\leqslant i\leqslant n-1}$ and relations,

$$(T_i - t)(T_i + 1) = 0, \qquad \text{(quaratic relation)}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad \text{(braid relation)}$$

$$T_i T_j = T_j T_i, \qquad \forall i, j, |i - j| > 1. \qquad \text{(commutativity)}$$

- Generators and inverses can be realized as operators on the space of polynomials $\mathbb{C}[z_1, \dots, z_n]$,

$$\begin{cases} T_i & = t - \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(1 - s_i), \qquad (\Rightarrow \mathbb{L}_i = T_i - t), \\ T_i^{-1} & = t^{-1} - t^{-1}\left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(1 - s_i). \end{cases}$$

mASEP on the segment: tKZ equation

- Consider a family of polynomials $\{f_\nu(z) : \nu = (\nu_1, \ldots, \nu_n)\}$ in $\mathbb{C}[z_1, \ldots, z_n]$.
- We say that $\{f_\nu(z)\}$ is a solution to the ASEP exchange relations if

$$T_i f_\nu = \begin{cases} f_{s_i\nu}, & \text{if } \nu_i > \nu_{i+1}, \\ t f_\nu, & \text{if } \nu_i = \nu_{i+1}, \end{cases} \tag{tKZ}$$

  for all ν and $1 \leqslant i \leqslant n-1$.
- Using the quadratic relation $T_i^2 + (1-t)T_i - t = 0$, (tKZ) gives,

$$T_i f_\nu = (t-1) f_\nu + t f_{s_i\nu} \qquad \text{if } \nu_i < \nu_{i+1}.$$

$$(s_i\nu)_i > (s_i\nu)_{i+1} \implies T_i f_{s_i\nu} = f_\nu \implies \underbrace{T_i^2}_{(t-1)T_i + t} f_{s_i\nu} = T_i f_\nu$$

$$\implies (t-1) f_\nu + t f_{s_i\nu} = T_i f_\nu.$$

□

mASEP on the segment: tKZ equation

- Consider a family of polynomials $\{f_\nu(z) : \nu = (\nu_1, \dots, \nu_n)\}$ in $\mathbb{C}[z_1, \dots, z_n]$.
- We say that $\{f_\nu(z)\}$ is a solution to the ASEP exchange relations if

$$T_i f_\nu = \begin{cases} f_{s_i\nu}, & \text{if } \nu_i > \nu_{i+1}, \\ t f_\nu, & \text{if } \nu_i = \nu_{i+1}, \end{cases} \tag{tKZ}$$

  for all ν and $1 \leqslant i \leqslant n-1$.
- Using the quadratic relation $T_i^2 + (1-t)T_i - t = 0$, (tKZ) gives,

$$T_i f_\nu = (t-1)f_\nu + t f_{s_i\nu} \qquad \text{if } \nu_i < \nu_{i+1}.$$

- Let $\mathbb{L}_i = T_i - t$. $\qquad \mathbb{L}_i f_\nu = \begin{cases} f_{s_i\nu} - t f_\nu, & \text{if } \nu_i > \nu_{i+1}, \\ 0, & \text{if } \nu_i = \nu_{i+1}, \\ t f_{s_i\nu} - f_\nu, & \text{if } \nu_i < \nu_{i+1}. \end{cases}$

  $\rightsquigarrow$ (Adm) is satisfied.

mASEP on the segment: tKZ equation

- Given a family of polynomials $\{f_\nu(z) : \nu = (\nu_1, \ldots, \nu_n)\}$ in $\mathbb{C}[z_1, \ldots, z_n]$.

$$T_i f_\nu = \begin{cases} f_{s_i \nu}, & \text{if } \nu_i > \nu_{i+1}, \\ t f_\nu, & \text{if } \nu_i = \nu_{i+1}, \end{cases} \qquad \text{(tKZ)}$$

for all ν and $1 \leqslant i \leqslant n-1$.

$$\Downarrow$$

$$\mathbb{L}_i \, |\nu\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) \, |\nu'\rangle \,, \qquad \forall \nu. \qquad \text{(Adm)}$$

- Look for $\{f_\nu(z) : \nu = (\nu_1, \ldots, \nu_n)\}$ satisfying tKZ.

From tKZ to duality

Prop. 3.2 : If $\{|\nu\rangle = f_\nu(z)\}$ satisfies (tKZ), then the function $\Psi(\nu,\mu) = \delta_{\nu,\mu}$ is a local mASEP duality function. In other words,

$$\mathbb{L}_i|\Psi\rangle = \mathbb{M}_i|\Psi\rangle, \quad 1 \leqslant i \leqslant n-1,$$

where $|\Psi\rangle = \sum_\nu \sum_\mu \Psi(\nu,\mu) f_\nu(z)|\mu\rangle = \sum_\mu f_\mu(z)|\mu\rangle$.

- Proof by direct computation.
- The trivial (diagonal) duality function is not interesting.
- Find a particular family of polynomials $\{f_\nu(z)\}$ satisfying (tKZ) and construct a non-trivial duality function.
- Key idea: construct functions $\{f_\nu(z)\}$ depending on additional parameters t and q and extract certain coefficients.

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Non-symmetric Macdonald Polynomials: Compositions

- A composition μ is an n-tuple of non-negative integers $(\mu_1, \ldots, \mu_n)$.
- A partition λ is a composition s.t. $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$.
- An anti-partition δ is a composition s.t. $0 \leqslant \delta_1 \leqslant \cdots \leqslant \delta_n$.
- For a given composition μ, write μ^+ and μ^- its unique partition / anti-partition obtained by permutation μ. $\rightsquigarrow$ Example: $(1, 0, 2)^+ = (2, 1, 0)$.
- Given two compositions μ and ν, define two orders:

$$\mu \geqslant \nu \iff \sum_{i=1}^{j} \mu_i \geqslant \sum_{i=1}^{j} \nu_i, \quad \forall 1 \leqslant j \leqslant n.$$

$$\mu \succ \nu \iff \mu^+ > \nu^+ \text{ or } \mu^+ = \nu^+, \mu > \nu.$$

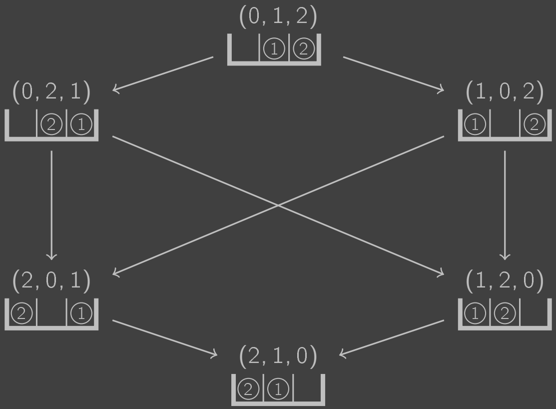
- Example: $(0,0) \prec \boxed{(0,1) \prec (1,0)} \prec (1,1) \prec \boxed{(0,2) \prec (2,0)}$.
- It is not a total order: $(2,2) \not\prec (3,0)$ and $(3,0) \not\prec (2,2)$

Non-symmetric Macdonald Polynomials: Compositions

For a composition μ, its composition sector $\sigma(\mu)$ is defined by

$$\sigma(\mu) := \{\nu \mid \mu^+ = \nu^+\} = S_n \cdot \mu.$$

Example: $n = 3$ with $\mu = (0, 1, 2)$.



Non-symmetric Macdonald Polynomials: Definition

- Multi-variate polynomial ring: $\mathbb{C}_{q,t}[z_1, \ldots, z_n] = \mathbb{C}(q, t)[z_1, \ldots, z_n]$ where t is the rate of ASEP and q is an additional parameter.
- Consider the Hecke algebra of type A_{n-1} as before. Define

$$(\omega g)(z_1, \ldots, z_n) = g(qz_n, z_1, \ldots, z_{n-1}).$$

- Define the Cherednik-Dunkl operators Y_i [Che91] [1] [Che95] [2],

$$Y_i = T_i \ldots T_{n-1} \omega T_1^{-1} \ldots T_{i-1}^{-1}, \quad 1 \leqslant i \leqslant n.$$

- Fact: the operators (Y_i) commute, so can be jointly diagonalized.
- Non-symmetric Macdonald Polynomials (NSMP) are defined as "normalized" eigenfunctions of these operators.
- NSMP are indexed by compositions $\mu \in \mathbb{Z}_{\geqslant 0}^n$.

[1] A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras (1991)
[2] Non-symmetric Macdonald's polynomials (1995)

Non-symmetric Macdonald Polynomials: Properties

- The change of basis w.r.t. the canonical basis is triangular:

$$E_\mu = z^\mu + \sum_{\nu \prec \mu} c_{\mu,\nu}(q,t) z^\nu, \quad c_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t).$$

- Eigenvalues are given by, for all $1 \leqslant i \leqslant n$,

$$Y_i E_\mu = y_i(\mu; q, t) E_\mu,$$
$$y_i(\mu; q, t) = q^{\mu_i} t^{\rho(\mu)_i + n - i + 1}, \quad \rho(\mu) = -w_\mu \cdot (1, 2, \ldots, n),$$

where $w_\mu \in S_n$ is the permutation with minimal length s.t. $\mu = w_\mu \cdot \mu^+$.

- Let μ be a composition such that $\mu_i < \mu_{i+1}$. Then

$$E_{s_i \mu} = t^{-1} \left(T_i + \frac{1 - t}{1 - y_{i+1}(\mu)/y_i(\mu)} \right) E_\mu.$$

Non-symmetric Macdonald Polynomials: Examples

- Example for $n = 2$.

$$E_{(0,0)} = 1$$

$$E_{(0,1)} = z_2$$

$$E_{(1,0)} = z_1 + \left(\frac{1-t}{1-qt}\right) z_2$$

$$E_{(1,1)} = z_1 z_2$$

$$E_{(0,2)} = \left(\frac{q(1-t)}{1-qt}\right) z_1 z_2 + z_2^2$$

$$E_{(2,0)} = z_1^2 + \left(\frac{q(1-t)^2}{(1-q^2t)(1-qt)} + \frac{1-t}{1-qt}\right) z_1 z_2 + \left(\frac{1-t}{1-q^2t}\right) z_2^2$$

- $(0,0) \prec \boxed{(0,1) \prec (1,0)} \prec (1,1) \prec \boxed{(0,2) \prec (2,0)}$.
- The order is defined in [CGW20] [1].

$$\mu \geqslant \nu \iff \sum_{i=1}^{j} \mu_i \geqslant \sum_{i=1}^{j} \nu_i, \quad \forall 1 \leqslant j \leqslant n.$$

$$\mu \succ \nu \iff \mu^+ > \nu^+ \text{ or } \mu^+ = \nu^+, \mu > \nu.$$

[1] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

Non-symmetric Macdonald Polynomials: Examples

- Example for $n = 2$.

$E_{(0,0)}$	$=$	1				
$E_{(1,0)}$	$=$	z_1				
$E_{(0,1)}$	$=$	$\left(\frac{1-t}{1-qt}\right) z_1$			$+z_2$	
$E_{(1,1)}$	$=$				$z_1 z_2$	
$E_{(2,0)}$	$=$	z_1^2	$+$	$\left(\frac{q(1-t)}{1-qt}\right)$	$z_1 z_2$	
$E_{(0,2)}$	$=$	$\left(\frac{1-t}{1-q^2 t}\right) z_1^2$	$+$	$\left(\frac{q(1-t)^2}{(1-q^2 t)(1-qt)} + \frac{1-t}{1-qt}\right)$	$z_1 z_2$	$+z_2^2$

- $(0,0) \prec \boxed{(1,0) \prec (0,1)} \prec (1,1) \prec \boxed{(2,0) \prec (0,2)}$.
- The order is defined in [HHL08] [1] also implemented in Sage.

$$\mu \geqslant \nu \iff \sum_{i=1}^{j} \mu_{n-i+1} \geqslant \sum_{i=1}^{j} \nu_{n-i+1}, \quad \forall 1 \leqslant j \leqslant n.$$

$$\mu \succ \nu \iff \mu^+ > \nu^+ \text{ or } \mu^+ = \nu^+, \mu > \nu.$$

[1] A combinatorial formula for nonsymmetric Macdonald polynomials (2008)

Non-symmetric Macdonald Polynomials: Coefficients

Conjecture 3.8 : Fix μ a composition. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$. If

$$\mathrm{Coeff}_p\left[E_\mu, m\right] := \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

exists and is non-zero. Then, there exists a unique ν for which

$$\mathrm{Coeff}_p\left[E_\mu, m\right] \propto E_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_\nu(z; q, t).$$

- One can check some examples using Sage.
- Authors of [CGW20] [1] are able to show this conjecture only in some particular cases which is enough to construct some non-trivial duality functions.

[1] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

Non-symmetric Macdonald Polynomials: Coefficients

Prop. 3.5 : Define generating series

$$Y(w) = \sum_{i=1}^{n} Y_i w^i, \qquad y_\mu(w) = \sum_{i=1}^{n} y_i(\mu; q, t) w^i.$$

Then,

$$E_\mu(z; q, t) = \prod_{\nu \prec \mu} \frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot z^\mu.$$

- This can be seen as "Lagrange interpolation".

Proof: Use the following two properties:

- For $\nu \prec \mu$, $\frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot E_\kappa = \begin{cases} 0, & \text{if } \kappa = \nu, \\ E_\mu, & \text{if } \kappa = \mu. \end{cases}$
- $E_\mu = z^\mu + \sum_{\nu \prec \mu} c_{\mu,\nu}(q,t) z^\nu \iff z^\mu = E_\mu + \sum_{\nu \prec \mu} d_{\mu,\nu}(q,t) E_\nu.$

□

Non-symmetric Macdonald Polynomials: Coefficients

Prop. 3.6 : Fix a composition μ. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$. If

$$\mathrm{Coeff}_p\,[E_\mu, m] := \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

exists and is non-zero. Then,

$$\mathrm{Coeff}_p\,[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p \sum_{\nu \in \mathcal{E}_\mu} c_\nu(q, t) E_\nu(z; q, t),$$

where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$.

Recall that the eigenvalues are $y_i(\mu; q, t) = q^{\mu_i} t^{\rho(\mu)_i + n - i + 1}$.

Thus,

$$\begin{aligned} y_\mu(w) = y_\nu(w) &\iff y_i(\mu) = y_i(\nu), && \forall 1 \leqslant i \leqslant n, \\ &\iff q^{\mu_i} t^{\rho(\mu)_i} = q^{\nu_i} t^{\rho(\nu)_i}, && \forall 1 \leqslant i \leqslant n. \end{aligned}$$

Unique solution $\nu = \mu$ for generic q and t; more solutions at $q = t^{-m}$.

Non-symmetric Macdonald Polynomials: Coefficients

Prop. 3.6 : Fix a composition μ. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$. If

$$\mathrm{Coeff}_p\,[E_\mu, m] := \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

exists and is non-zero. Then,

$$\mathrm{Coeff}_p\,[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p \sum_{\nu \in \mathcal{E}_\mu} c_\nu(q, t) E_\nu(z; q, t),$$

where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$.

Proof: Use “Lagrange interpolation”,

$$\prod_{\substack{\nu \prec \mu \\ \nu \notin \mathcal{E}_\nu}} \frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot z^\mu = E_\mu + \sum_{\nu \in \mathcal{E}_\mu} d_{\mu,\nu}(q, t) E_\nu.$$

By taking $\lim_{q \to t^{-m}} (1 - qt^m)^p$ on both sides, LHS = 0 and RHS gives the proposition. □

Non-symmetric Macdonald Polynomials: Coefficients

Prop. 3.7 : Assume that $p = |\mathcal{E}_\mu|$, then there exists a unique ν for which

$$E_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_\nu(z; q, t)$$

is well-defined and $\mathrm{Coeff}_p [E_\mu, m] \propto E_\nu(z; t^{-m}, t)$.

- Conjecture 3.8 is true with an additional condition $p = |\mathcal{E}_\mu|$.
- Proof uses "Lagrange interpolation" and by induction.
- The composition ν corresponds to the minimal composition in $\mathcal{E}_\mu$.

Non-symmetric Macdonald Polynomials: Coefficients

- Example of $\mathcal{E}_\mu$ for $\mu = (0,0,2)$ and $m = 1$.
- w_μ is the permutation with minimal length s.t. $\mu = w_\mu \cdot \mu^+$.

  $\rightsquigarrow w_\mu = (13)$ since the permutation (312) is longer than (13).
- Define $S_m(\mu) := (m\mu_i - \rho(\mu)_i)_i = m(0,0,2) + (3,2,1) = (3,2,2m+1)$. Then,

$$\nu \in \mathcal{E}_\mu \iff S_m(\mu) = S_m(\nu).$$

- $\sum \mu_i = \sum \nu_i \Rightarrow \nu$ must contain at least one 0.
- Possible candidates for ν are s.t. $\nu^- = (0,1,1)$ or $(0,0,2)$.
  - If ν is s.t. $\nu^- = (0,1,1)$, then $\nu \prec \mu$ and $S_1(\mu) = S_1(\nu)$.
  - $(0,0,2)$ is a trivial solution. Moreover, its permutations are not smaller than μ for the partial order $\prec$.

Non-symmetric Macdonald Polynomials: Coefficients

- Example of $\mathcal{E}_\mu$ for $\mu = (0,0,2)$ and $m = 1$.
- w_μ is the permutation with minimal length s.t. $\mu = w_\mu \cdot \mu^+$.

  $\rightsquigarrow w_\mu = (13)$ since the permutation (312) is longer than (13).
- Define $S_m(\mu) := (m\mu_i - \rho(\mu)_i)_i = m(0,0,2) + (3,2,1) = (3,2,2m+1)$. Then,

$$\nu \in \mathcal{E}_\mu \iff S_m(\mu) = S_m(\nu).$$

- $\sum \mu_i = \sum \nu_i \Rightarrow \nu$ must contain at least one 0.
- Possible candidates for ν are s.t. $\nu^- = (0,1,1)$ or $(0,0,2)$.
- Conclusion: $\mathcal{E}_{(0,0,2)} = \sigma((0,1,1)) := S_3 \cdot (0,1,1)$ for $m = 1$.
- Similarly, one can show that:

  If $\delta = (0^{n-m}, r^m)$, then $\mathcal{E}_\delta = \sigma(\varepsilon)$ where $\varepsilon = (0^{n-rm}, 1^{rm})$.

Outline

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- Matrix product Ansatz and examples.
- Global duality.

ASEP polynomials: Definition

- Another basis ASEP polynomials can be defined from NSMP,

$$f_\delta(z;q,t) = E_\delta(z;q,t), \qquad \forall \delta = (\delta_1 \leqslant \ldots \leqslant \delta_n),$$
$$f_{s_i\mu} = T_i^{-1} f_\mu, \qquad \mu_i < \mu_{i+1}.$$

- Well-defined due to the Hecke algebra structure.
- $f_\mu \neq E_\mu$ in general: the recursion formula is different.

> Recall that $E_{s_i\mu} = t^{-1}\left(T_i + \frac{1-t}{1 - y_{i+1}(\mu)/y_i(\mu)}\right) E_\mu$ for $\mu_i < \mu_{i+1}$.

ASEP polynomials: Definition

- Another basis ASEP polynomials can be defined from NSMP,

$$f_\delta(z;q,t) = E_\delta(z;q,t), \qquad \forall \delta = (\delta_1 \leqslant \ldots \leqslant \delta_n),$$
$$f_{s_i\mu} = T_i^{-1} f_\mu, \qquad \mu_i < \mu_{i+1}.$$

- Well-defined due to the Hecke algebra structure.
- $f_\mu \neq E_\mu$ in general: the recursion formula is different.
- ASEP polynomials are triangular w.r.t. the canonical basis,

$$f_\mu = z^\mu + \sum_{\nu \prec \mu} c_{\mu,\nu}(q,t) z^\nu,$$

  and satisfies (tKZ).
- $$f_{\mu_n,\mu_1,\ldots,\mu_{n-1}}(qz_n, z_1, \ldots, z_{n-1}; q, t) = q^{\mu_n} f_\mu(z;q,t). \qquad \text{(Cyclic-B.C.)}$$
- Above two properties uniquely define ASEP polynomials.
- ASEP and NSMP are also related by a triangular change of basis.

ASEP polynomials: Properties

Prop. 3.10 : For any composition μ, the following expansions are unique,

$$E_\mu(z;q,t) = f_\mu(z;q,t) + \sum_{\substack{\nu\in\sigma(\mu)\\ \nu\prec\mu}} c_{\mu,\nu}(q,t) f_\nu(z;q,t);$$

$$f_\mu(z;q,t) = E_\mu(z;q,t) + \sum_{\substack{\nu\in\sigma(\mu)\\ \nu\prec\mu}} d_{\mu,\nu}(q,t) E_\nu(z;q,t).$$

Proof: For an anti-partition, $E_\delta = f_\delta$ so the proposition is true. By induction, assume μ is s.t. the proposition holds with $\mu_i < \mu_{i+1}$,

$$E_{s_i\mu} = t^{-1}\left(T_i + \frac{1-t}{1-y_{i+1}(\mu)/y_i(\mu)}\right)\left(f_\mu + \sum_{\substack{\nu\in\sigma(\mu)\\ \nu\prec\mu}} c_{\mu,\nu}(q,t) f_\nu\right).$$

- $T_i f_\nu = (t-1)f_\nu + t f_{s_i\nu}$ if $\nu_i < \nu_{i+1}$;
- $\nu, s_i\nu \in \sigma(\mu) = \sigma(s_i\mu)$;
- $(\nu \prec \mu, \mu \prec s_i\mu) \Rightarrow \nu, s_i\nu \prec s_i\mu$.

ASEP polynomials: Properties

Thm 3.11 : Fix an anti-partition δ. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$ such that Conjecture 3.8 holds for δ. Then, there exists a unique anti-partition ε,

$$f_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} f_\nu(z; q, t)$$

is well-defined for all $\nu \in \sigma(\varepsilon)$ and

$$\mathrm{Coeff}_p\,[f_\mu, m] = \sum_{\nu \in \sigma(\varepsilon)} \Psi(\nu, \mu; t) f_\nu(z; t^{-m}, t)$$

for all $\mu \in \sigma(\delta)$ and suitable coefficients $\Psi(\nu, \mu; t)$.

Proof:

- The proof is based on Prop. 3.10.
- Use the recurrence relation and the fact that T_i commutes with lim.

Theorem 3.12 : Keep the notations from Theorem 3.11. $\Psi(\nu, \mu; t)$ defines a local duality function.

Proof: From Prop 3.2 (trivial duality function), the following holds

$$\mathbb{L}_i|\mathcal{I}\rangle = \mathbb{M}_i|\mathcal{I}\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{1}$$

where $|\mathcal{I}\rangle = \sum_{\mu} f_\mu(z; q, t)|\mu\rangle$. Taking the coefficient,

$$|\mathcal{I}_{p,m}\rangle := \mathrm{Coeff}_p\,[|\mathcal{I}\rangle, m] = \sum_{\mu\in\sigma(\delta)} \sum_{\nu\in\sigma(\varepsilon)} \Psi(\nu, \mu; t) f_\nu(z; t^{-m}, t)|\mu\rangle$$

also satisfies $\mathbb{L}_i|\mathcal{I}_{p,m}\rangle = \mathbb{M}_i|\mathcal{I}_{p,m}\rangle, \quad 1 \leqslant i \leqslant n-1.$

Outline

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Matrix product Ansatz

- The coefficients $\Psi(\nu,\mu)$ are difficult to compute in general.
- The matrix product Ansatz from [CGW15] [1] is useful for ASEP polynomials.

Theorem : Fix $r \geqslant 1$. For a composition μ with largest part equal to r,

$$f_\mu(z;q,t) = \Omega_\mu(q,t) \times \mathrm{Tr}\Big(A_{\mu_1}(z_1)\dots A_{\mu_n}(z_n)S\Big), \qquad \text{(Mat-Ans)}$$

where $\{A_i(z)\}_{0\leqslant i\leqslant r}$ and S are explicit matrices.

- Idea : translate (tKZ) and (Cyclic-B.C.) into matrices.

[1] Matrix product formula for Macdonald polynomials (2015)

Matrix product Ansatz

Theorem : Fix $r \geqslant 1$. For a composition μ with largest part equal to r,

$$f_\mu(z;q,t) = \Omega_\mu(q,t) \times \mathrm{Tr}\Big(A_{\mu_1}(z_1)\dots A_{\mu_n}(z_n) S\Big), \qquad \text{(Mat-Ans)}$$

where $\{A_i(z)\}_{0 \leqslant i \leqslant r}$ and S are explicit matrices.

- Matrices involved in (Mat-Ans) are infinite-dimensional matrices ϕ, $\phi^\dagger$, k,

$$[\phi]_{i,j} = \delta_{i+1,j}(1-t^i), \qquad [\phi^\dagger]_{i,j} = \delta_{i,j+1}, \qquad [k]_{i,j} = \delta_{i,j} t^i, \qquad i,j \in \mathbb{N}.$$

- For $r = 1$,

  $A_0(z) = 1,$

  $A_1(z) = z.$

- For $r = 2$,

  $A_0(z) = 1 + z\phi,$

  $A_1(z) = zk,$

  $A_2(z) = z\phi^\dagger + z^2.$

Matrix product Ansatz: $\mu^- = (0^{n-m}, r^m)$

Thm 5.5 : Let $r, m \in \mathbb{N}$ such that $n - rm \geqslant 0$. Take $\delta = (0^{n-m}, r^m)$. Then, $\mathrm{Coeff}_1[f_\delta, m]$ exists and

$$\mathrm{Coeff}_1[f_\mu, m] = \sum_{\nu \in \sigma(\varepsilon)} \psi(\nu, \mu; t) z^\nu, \quad \forall \mu \in \sigma(\delta),$$

where $\varepsilon = (0^{n-rm}, 1^{rm})$.

Proof:

- Check $\mathrm{Coeff}_1[f_\delta, m]$ exists using the explicit formula.
- $$\mathrm{Coeff}_p[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p \sum_{\nu \in \mathcal{E}_\mu} c_\nu(q, t) E_\nu(z; q, t),$$

  where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$ contains only rank-one compositions.
- Use the fact that for any rank-one composition ν, f_ν is multilinear in variables.

Matrix product Ansatz: $\mu^- = (0^{n-m}, r^m)$

Thm 5.6 : We have that, $\Psi(\nu,\mu) = d(t)\cdot t^{\Omega(\mu,\nu)}\cdot I(\mu,\nu)$, where

- $\Omega(\mu,\nu) = \sum_{1\leqslant i<j\leqslant n} \mathbb{1}(\mu_i < \mu_j)\mathbb{1}(\nu_i = \nu_j = 1)$.
- $I(\mu,\nu) = \begin{cases} 0, & \exists k : (\mu_k,\nu_k) = (r,0), \\ 1, & \text{otherwise.} \end{cases}$

Proof: Check for $\mu = \delta^+ = (r^m, 0^{n-m})$ then by induction.

- $A_r(z)$ can be factorized by $z \Longrightarrow f_{\delta+}$ contains the factor $\prod_{i=1}^m z_i$.
- $f_{\delta+}$ is homogeneous in $(z1,\ldots,z_n)$ of degree rm.
- $f_{\delta+}$ is symmetric in $z_{m+1},\ldots,z_n$.
- Thm 5.5: $f_{\delta+}$ is multilinear in all the variables.

Matrix product Ansatz: $\mu^- = (0^{n-m}, r^m)$

Thm 5.6 : We have that, $\Psi(\nu, \mu) = d(t) \cdot t^{\Omega(\mu,\nu)} \cdot I(\mu, \nu)$, where

- $\Omega(\mu, \nu) = \sum_{1 \leqslant i < j \leqslant n} \mathbb{1}(\mu_i < \mu_j)\mathbb{1}(\nu_i = \nu_j = 1)$.
- $I(\mu, \nu) = \begin{cases} 0, & \exists k : (\mu_k, \nu_k) = (r, 0), \\ 1, & \text{otherwise.} \end{cases}$

Proof: Check for $\mu = \delta^+ = (r^m, 0^{n-m})$ then by induction.

$$\Rightarrow \quad \mathrm{Coeff}_1\left[f_{\delta^+}, m\right] = d(t) \cdot \prod_{i=1}^{m} z_i \cdot e_{(rm-n)}(z_{m+1}, \ldots, z_n)$$
$$= d(t) \sum_{\nu \in \sigma(\varepsilon)} I(\delta^+, \nu) z^\nu,$$

since $\Omega(\mu, \nu) = 0$ for all ν. □

Matrix product Ansatz: $\mu^- = (0^{n-m}, 1^m)$

- Thm 5.6 recovers the duality function from [BCS+14] [1].

Thm 5.1 : Keeping the same notations, we have,

$$t^{\Omega(\mu,\nu)} \cdot I(\mu,\nu) = t^{-m(m-1)/2} \prod_{j=1}^{m} \Big(\prod_{1 \leqslant i < x_j(\mu)} t^{\nu_i} \Big) \nu_{x_j(\mu)}.$$

Proof:

- If $I(\mu,\nu) = 0$, then LHS = RHS = 0.
- If $I(\mu,\nu) = 1$, i.e. $\forall k, \mu_k \leqslant \nu_k$.
  The set $A := \{(i,j) : i < j, \nu_i = 1, \mu_j = 1\}$ can be partitioned into:
  - $A_1 := \{(i,j) : i < j, \mu_i = \mu_j = 1\}$; $|A_1| = \frac{1}{2}m(m-1)$.
  - $A_2 := \{(i,j) : i < j, \mu_i = 0, \mu_j = 1\}$; $\Omega(\mu,\nu) = |A_2|$.

[1] From duality to determinants for q-TASEP and ASEP (2014)

Matrix product Ansatz: $\mu^- = (0^{n-m}, 1^m)$

- Thm 5.6 recovers the duality function from [BCS+14] [1].

Thm 5.1 : Keeping the same notations, we have,

$$t^{\Omega(\mu,\nu)} \cdot I(\mu,\nu) = t^{-m(m-1)/2} \prod_{j=1}^{m} \Big(\prod_{1 \leqslant i < x_j(\mu)} t^{\nu_i} \Big) \nu_{x_j(\mu)}.$$

Proof:

- If $I(\mu,\nu) = 0$, then LHS = RHS = 0.
- If $I(\mu,\nu) = 1$, i.e. $\forall k, \mu_k \leqslant \nu_k$.

$$\Longrightarrow \sum_{j=1}^{m} \sum_{i=1}^{x_j(\mu)-1} \nu_i = \Omega(\mu,\nu) + \frac{1}{2} m(m-1).$$

□

[1] From duality to determinants for q-TASEP and ASEP (2014)

Matrix product Ansatz: $\mu^- = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2})$

Prop. 6.4 & Cor. 6.5 : Let μ be a rank-two decomposition with

$$\vec{x}(\mu) = (x_1 < \cdots < x_{m_1}),$$
$$\vec{y}(\mu) = (y_1 < \cdots < y_{m_2}),$$

labelling the positions of 1 and 2-particles. Then,

$$\Psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \Big(\prod_{i<x} t^{\nu_i} \Big)^{\nu_x} \cdot \prod_{y \in \vec{y}(\mu)} \Big(\prod_{i<y} t^{\nu_i} \Big)^{\nu_y} \cdot t^{-\chi(\vec{x}, \vec{y})}$$

is a (local) duality function, where

$$\chi(\vec{x}, \vec{y}) := \#\{(x_i, y_j) \in (\vec{x}, \vec{y}) : x_i > y_j\}.$$

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Global duality: Rank-one

Prop. 7.1 : Let ν be an infinite rank-one composition. Let μ also be a rank-one composition and denote its particle positions to be $\vec{x}(\mu) = (x_1 < \cdots < x_m)$. Then the observable

$$H(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \left(\prod_{i \leqslant x} t^{\nu_i} \right)$$

is a (global) duality function.

- It is not a local duality function.
- Given in [BCS+14] [1].
- No constructive proof. Can be checked once the formula guessed.

[1] From duality to determinants for q-TASEP and ASEP (2014)

Global duality: Rank-two

Prop. 7.2 : Let ν be an infinite rank-one composition. Let μ be a rank-two composition with 1 and 2-particle positions given by,

$$\vec{x}(\mu) = (x_1 < \cdots < x_{m_1}),$$
$$\vec{y}(\mu) = (y_1 < \cdots < y_{m_2}),$$

Then the observable

$$H(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \Big(\prod_{i \leqslant x} t^{\nu_i} \Big) \cdot \prod_{y \in \vec{y}(\mu)} \Big(\prod_{i \leqslant y} t^{\nu_i} \Big) \cdot t^{-\chi(\vec{x}, \vec{y})}$$

is a (global) duality function, where

$$\chi(\vec{x}, \vec{y}) := \#\{(x_i, y_j) \in (\vec{x}, \vec{y}) : x_i > y_j\}.$$

- Direct check using the rank-one global duality Prop. 7.1.
- Proof by looking at blocks of particles + color-blind argument (rank-one composition).

Cor. 7.3 : Let ν be an infinite rank-one composition. Let μ be a rank-r composition with positions of j-particles ($1 \leqslant j \leqslant r$) labelled by,

$$\vec{x}^{(j)}(\mu) = (x_1^{(j)} < \cdots < x_{m_j}^{(j)}).$$

Then the observable

$$H(\nu,\mu) = \prod_{j=1}^{r} \left(\prod_{x \in \vec{x}^{(j)}} \left(\prod_{i \leqslant x} t^{\nu_i} \right) \right) t^{-\chi(\vec{x}^{(1)},\ldots,\vec{x}^{(r)})}$$

is a (global) duality function, where

$$\chi(\vec{x}^{(1)},\ldots,\vec{x}^{(r)}) := \#\Big\{ (x,y) \in (\vec{x}^{(i)}, \vec{y}^{(j)}) : i < j, x > y \Big\}.$$

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