# Integrable systems and duality

Based on Chen, de Gier, Wheeler

"Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation"

# Outline

- The notion of the duality: operator / matrix formalism.
- Multi-species ASEP process.
- Hecke algebra, tKZ equation and trivial duality function.
- Non-symmetric Macdonald polynomials.
- ASEP polynomials: construction of non-trivial duality functions.
- Matrix product Ansatz and examples.

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## Duality for Markov processes: Definition

- Given two Markov processes on state spaces $A$ and $B$,

$$X = (\Omega_A, \mathcal{F}_A, (X_t)_{t\geqslant 0}, \{\mathbb{P}_a\}_{a\in A}), \qquad Y = (\Omega_B, \mathcal{F}_B, (Y_t)_{t\geqslant 0}, \{\mathbb{P}^b\}_{b\in B}).$$

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- A function $\Psi : A \times B \to \mathbb{C}$ is a duality function for $X$ and $Y$ iff:

$$\forall a \in A, b \in B, \forall t \geqslant 0, \quad \mathbb{E}_a[\Psi(X_t, b)] = \mathbb{E}^b[\Psi(a, Y_t)].$$

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- Let $(P_t)$ and $(Q_t)$ be semi-groups of $(X_t)$ and $(Y_t)$. Recall: $P_t f(a) = \mathbb{E}_a[f(X_t)]$.

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- Write

$$\begin{aligned} L[\Psi(\cdot, b)](a) &= \sum_{a'\in A} \ell(a, a')\Psi(a', b), \\ M[\Psi(a, \cdot)](b) &= \sum_{b'\in B} m(b, b')\Psi(a, b'). \end{aligned} \tag{Gen}$$

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$\Psi : A \times B \to \mathbb{C}$ is a duality function.

$$\Updownarrow$$

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where

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## Duality for Markov processes: Matrix formalism

- Consider vector spaces with elements of $A$ and $B$ as bases,

$$\mathbb{A} = \text{Span}\left\{|a\rangle, a \in A\right\}, \qquad \mathbb{B} = \text{Span}\left\{|b\rangle, b \in B\right\}.$$

- Define an element in the tensor product $|\Psi\rangle \in \mathbb{A} \otimes \mathbb{B}$,

$$|\Psi\rangle = \sum_{\substack{a \in A \\ b \in B}} \Psi(a, b) |a\rangle \otimes |b\rangle.$$

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$$|\Psi\rangle = \sum_{\substack{a\in A \\ b \in B}} \Psi(a,b)|a\rangle \otimes |b\rangle.$$

- Define linear operators $\mathbb{L} \in \mathrm{End}(\mathbb{A})$ and $\mathbb{M} \in \mathrm{End}(\mathbb{A})$,

$$\mathbb{L}\,|a\rangle = \sum_{a' \in A} \ell(a',a)\,|a'\rangle, \qquad \mathbb{M}\,|b\rangle = \sum_{b' \in B} m(b',b)\,|b'\rangle,$$

where coefficients $\ell(a',a)$ and $m(a',a)$ are given by (Gen).

- $\mathbb{L}$ and $\mathbb{M}$ are right actions.

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Proof: Direct computation.

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$$= \sum_{\substack{a\in A \\ b\in B}} \sum_{a'\in A} \ell(a, a')\Psi(a', b) \; |a\rangle \otimes |b\rangle \, .$$

$$\mathbb{M}|\Psi\rangle = \sum_{b'\in A} \sum_{\substack{a\in A \\ b\in B}} m(b', b)\Psi(a, b) \, |a\rangle \otimes \left|b'\right\rangle$$

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## mASEP on the segment

The multi-species asymmetric simple exclusion process (mASEP):

- On the segment $\{1, \ldots, n\}$.
- At most one particle per site.
- Particles of type $1, \ldots, r$. Empty site is of type 0.
- Higher-type particle can switch with a lower-type particle.
- Switch rates are: $\tau_r$ to the right, $\tau_\ell$ to the left. For $i > j$,


- ASEP $\Leftrightarrow r = 1$.

# mASEP on the segment

A configuration can be encoded in two ways.

Monomial: $z^\nu \in \mathbb{C}[z_1, \ldots, z_n]$.

- $A = \left\{ z^\nu := \prod_{i=1}^{n} z_i^{\nu_i}, 0 \leqslant \nu_i \leqslant r \right\}.$

Sequence : $\mu = (\mu_i)_{1 \leqslant i \leqslant n} \in \{0, \ldots, r\}^n$.

- $B = \left\{ |\mu_1 \ldots \mu_n\rangle := \bigotimes_{i=1}^{n} |\mu_i\rangle_i, 0 \leqslant \mu_i \leqslant r \right\}$

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Particle content: $0^{m_0} 1^{m_1} \dots r^{m_r}$, $m_i = |\{k : \mu_k = i\}|$.

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A configuration can be encoded in two ways.

Example for $n = 5$:

Monomial: $z^\nu \in \mathbb{C}[z_1, \dots, z_n]$. $\rightsquigarrow$ Example: $z_2 z_3^2 z_5$.

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- $A = \left\{ z^\nu := \prod_{i=1}^{n} z_i^{\nu_i}, 0 \leqslant \nu_i \leqslant r \right\}$.
- $\mathbb{A} = \{P \in \mathbb{C}[z_1, \ldots, z_n]$ s.t. the degree in each variable is at most $r\}$.

Sequence : $\mu = (\mu_i)_{1 \leqslant i \leqslant n} \in \{0, \ldots, r\}^n$. $\rightsquigarrow$ Example: $|0, 1, 2, 0, 1\rangle$.

- $B = \left\{ |\mu_1 \ldots \mu_n\rangle := \bigotimes_{i=1}^{n} |\mu_i\rangle_i , 0 \leqslant \mu_i \leqslant r \right\}$
- $\mathbb{B} = \bigotimes_{i=1}^{n} V_i$ where $V_i \cong \mathbb{C}^{r+1}$ and $\{|0\rangle, \ldots, |r\rangle\}$ = canonical basis of $\mathbb{C}^{r+1}$.

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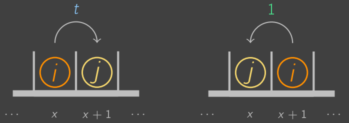
## mASEP on the segment: Local generator

- Fix $t > 0$.
- Consider two processes:
  - $\tau_r = t$ and $\tau_\ell = 1$:

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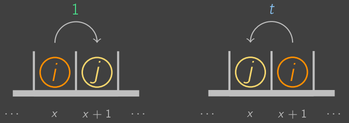


for $i > j$.

- The forward dynamics of one is the backward dynamics of the other.
- Denote their generators by $\mathbb{L}$ and $\mathbb{M}$.
- Initial conditions might be different!

## mASEP on the segment: Local generator

- The local generators $\mathbb{L}_i$ and $\mathbb{M}_i$ act on particles at positions $i$ and $i+1$.

$$\mathbb{L} = \sum_{i=1}^{n-1} \mathbb{L}_i, \qquad \mathbb{M} = \sum_{i=1}^{n-1} \mathbb{M}_i.$$

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- The local operators $L_i$ and $M_i$ are,

$$L_i[f](\nu) = \sum_{\nu'} \underbrace{\ell_i(\nu,\nu')}_{\tau(\nu\to\nu')} f(\nu'), \qquad M_i[f](\mu) = \sum_{\mu'} \underbrace{m_i(\mu,\mu')}_{\tau(\mu\to\mu')} f(\mu').$$

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- Example for ASEP ($r=1$):

$$L_i = (\ell_i(\nu,\nu'))_{\nu,\nu'} \qquad M_i = (m_i(\mu,\mu'))_{\mu,\mu'}$$

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & 0 & 0 \\ 01 & 0 & -1 & 1 & 0 \\ 10 & 0 & t & -t & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & 0 & 0 \\ 01 & 0 & -t & t & 0 \\ 10 & 0 & 1 & -1 & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array}$$

## mASEP on the segment: $r = 1$, ASEP

- Define $\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)$, where $s_i$ acts on polynomials by $z_i \leftrightarrow z_{i+1}$.

**Prop. 2.1** : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i\,|\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu)\,|\nu'\rangle$.

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Proof:

- $\mathbb{L}_i\,|1\rangle = \mathbb{L}_i\,|z_i z_{i+1}\rangle = 0$ due to the symmetry.
- $\mathbb{L}_i|z_i\rangle = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(z_{i+1} - z_i) = z_{i+1} - tz_i$.


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$L_i = (\ell_i(\nu, \nu'))_{\nu, \nu'}$

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
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| 01 | 0 | −1 | 1 | 0 |
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$$\mathbb{L}_i|\Psi\rangle = \mathbb{M}_i|\Psi\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{LD}$$

where

$$|\Psi\rangle = \sum_{\nu\in A}\sum_{\mu\in B}\Psi(\nu,\mu)\prod_{k=1}^{n} z_k^{\nu_k}|\mu\rangle.$$

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- Local duality $\Rightarrow$ global duality. (The reverse is false in general.)
- We focus on the local duality, i.e. solutions of (LD).

## mASEP on the segment: Local duality

- The following duality functions for ASEP are constructed in [BCS+14] [1].

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**Theorem 4.1 & 4.2** [Duality functions of ASEP on $\mathbb{Z}$]:

- ASEP occupation process $(\nu_x)_{x\in\mathbb{Z}}$ with rates $(\tau_r, \tau_\ell) = (1, t)$ ;
- Position process $\overrightarrow{x}(\mu) = (x_1(\mu) < \cdots < x_n(\mu))$ with rates $(\tau_r, \tau_\ell) = (t, 1)$.

$$\Psi(\nu, \mu) = \prod_{x\in\overrightarrow{x}(\mu)} \left(\prod_{i<x} t^{\nu_i}\right) \nu_x,$$

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- The first one can be recovered using the framework of [2] but not the second one.

[1] From duality to determinants for q-TASEP and ASEP (2014)

[2] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

## mASEP on the segment: Something wrong...

- Define $\mathbb{L}_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)$, where $s_i$ acts on polynomials by $z_i \leftrightarrow z_{i+1}$.

**Prop. 2.1** : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i\,|\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu)\,|\nu'\rangle$.

- $\mathbb{L}_i$ acts on polynomials faithfully for $r = 1$.

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$$\mathbb{L}_i\left|z_i^2\right\rangle = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1)\cdot z_i^2 = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(z_{i+1}^2 - z_i^2)$$

$$= z_{i+1}^2 + (1-t)z_i z_{i+1} - t z_i^2.$$

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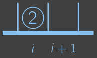
- For $r \geqslant 2$, consider the local dynamics starting from 

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- But the dynamics gives 

$\rightsquigarrow$ $1\cdot$ 

$- \, t\cdot$ 

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**Prop. 2.1** : Let $|\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i}$. Then, $\mathbb{L}_i |\nu\rangle = \sum_{\nu' \in \mathbb{A}} \ell_i(\nu', \nu) |\nu'\rangle$.

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- $\mathbb{L}_i$ acts on polynomials faithfully for $r = 1$.
- Conclusion: the basis $\left\{ |\nu\rangle = \prod_{i=1}^{n} z_i^{\nu_i} \right\}$ is not a good basis to work with.
- We look for a basis of polynomials $\{|\nu\rangle = f_\nu(z)\}$ such that

$$\mathbb{L}_i |\nu\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) \left|\nu'\right\rangle, \qquad \forall \nu. \qquad \text{(Adm)}$$

We say that $\{|\nu\rangle = f_\nu(z)\}$ is admissible.

# Outline

- The notion of the duality: operator / matrix formalism.
- Multi-species ASEP process.
- Hecke algebra, tKZ equation and trivial duality function.
- Non-symmetric Macdonald polynomials.
- ASEP polynomials: construction of non-trivial duality functions.
- Matrix product Ansatz and examples.

# mASEP on the segment: Hecke algebra

- Type $A_{n-1}$ Hecke algebra with generators $\{T_i\}_{1\leqslant i\leqslant n-1}$ and relations,

$$(T_i - t)(T_i + 1) = 0, \qquad \text{(quaratic relation)}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad \text{(braid relation)}$$

$$T_i T_j = T_j T_i, \qquad \forall i, j, |i-j| > 1. \qquad \text{(commutativity)}$$

- Generators and inverses can be realized as operators on the space of polynomials $\mathbb{C}[z_1, \ldots, z_n]$,

$$\begin{cases} T_i &= t - \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(1 - s_i), \qquad (\Rightarrow \mathbb{L}_i = T_i - t), \\ T_i^{-1} &= t^{-1} - t^{-1}\left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(1 - s_i). \end{cases}$$

## mASEP on the segment: tKZ equation

- Consider a family of polynomials $\{f_\nu(z) : \nu = (\nu_1, \dots, \nu_n)\}$ in $\mathbb{C}[z_1, \dots, z_n]$.
- We say that $\{f_\nu(z)\}$ is a solution to the ASEP exchange relations if

$$T_i f_\nu = \begin{cases} f_{s_i \nu}, & \text{if } \nu_i > \nu_{i+1}, \\ t f_\nu, & \text{if } \nu_i = \nu_{i+1}, \end{cases} \tag{tKZ}$$

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  for all $\nu$ and $1 \leqslant i \leqslant n-1$.
- Using the quadratic relation $T_i^2 + (1-t)T_i - t = 0$, (tKZ) gives,

$$T_i f_\nu = (t-1) f_\nu + t f_{s_i \nu} \qquad \text{if } \nu_i < \nu_{i+1}.$$

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$$\Downarrow$$

$$\mathbb{L}_i \left|\nu\right\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) \left|\nu'\right\rangle, \qquad \forall \nu. \qquad \text{(Adm)}$$

- Look for $\{f_\nu(z) : \nu = (\nu_1, \ldots, \nu_n)\}$ satisfying tKZ.

**Prop. 3.2** : If $\{|\nu\rangle = f_\nu(z)\}$ satisfies (tKZ), then the function $\Psi(\nu, \mu) = \delta_{\nu,\mu}$ is a local mASEP duality function. In other words,

$$\mathbb{L}_i|\Psi\rangle = \mathbb{M}_i|\Psi\rangle, \quad 1 \leqslant i \leqslant n-1,$$

where $|\Psi\rangle = \sum_\nu \sum_\mu \Psi(\nu, \mu) f_\nu(z)|\mu\rangle = \sum_\mu f_\mu(z)|\mu\rangle$.

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- Proof by direct computation.
- The trivial (diagonal) duality function is not interesting.
- Find a particular family of polynomials $\{f_\nu(z)\}$ satisfying (tKZ) and construct a non-trivial duality function.
- Key idea: construct functions $\{f_\nu(z)\}$ depending on additional parameters $t$ and $q$ and extract certain coefficients.

# Outline

- The notion of the duality: operator / matrix formalism.
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## Non-symmetric Macdonald Polynomials: Compositions

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- Example: $(0, 0) \prec \boxed{(0, 1) \prec (1, 0)} \prec (1, 1) \prec \boxed{(0, 2) \prec (2, 0)}$.
- It is not a total order: $(2, 2) \not\prec (3, 0)$ and $(3, 0) \not\prec (2, 2)$

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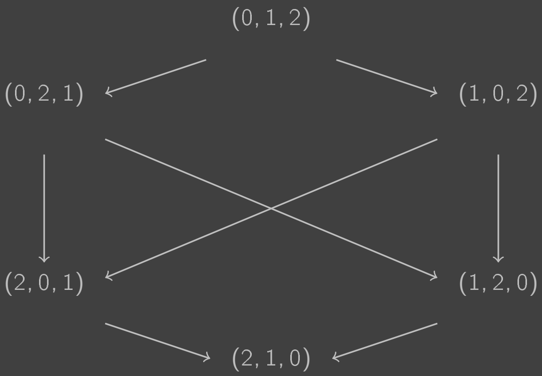

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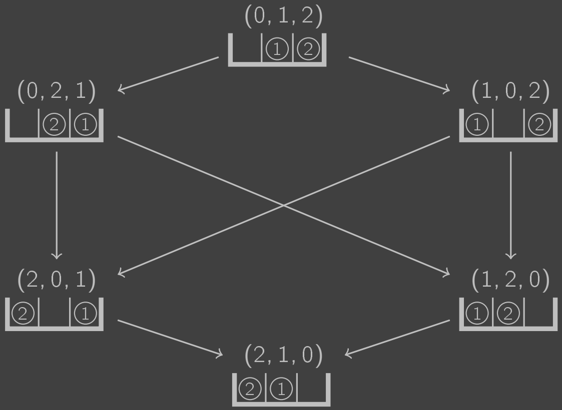

## Non-symmetric Macdonald Polynomials: Definition

- Multi-variate polynomial ring: $\mathbb{C}_{q,t}[z_1, \ldots, z_n] = \mathbb{C}(q, t)[z_1, \ldots, z_n]$ where $t$ is the rate of ASEP and $q$ is an additional parameter.

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- Define the Cherednik-Dunkl operators $Y_i$ [Che91] [1] [Che95] [2],

$$Y_i = T_i \ldots T_{n-1} \omega T_1^{-1} \ldots T_{i-1}^{-1}, \quad 1 \leqslant i \leqslant n.$$

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- Fact: the operators $(Y_i)$ commute, so can be jointly diagonalized.
- Non-symmetric Macdonald Polynomials (NSMP) are defined as "normalized" eigenfunctions of these operators.
- NSMP are indexed by compositions $\mu \in \mathbb{Z}_{\geqslant 0}^n$.

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# Non-symmetric Macdonald Polynomials: Properties

- The change of basis w.r.t. the canonical basis is triangular:

$$E_\mu = z^\mu + \sum_{\nu \prec \mu} c_{\mu,\nu}(q,t) z^\nu, \quad c_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t).$$

- Eigenvalues are given by, for all $1 \leqslant i \leqslant n$,

$$Y_i E_\mu = y_i(\mu; q, t) E_\mu,$$
$$y_i(\mu; q, t) = q^{\mu_i} t^{\rho(\mu)_i + n - i + 1}, \quad \rho(\mu) = -w_\mu \cdot (1, 2, \ldots, n),$$

  where $w_\mu \in S_n$ is the permutation with minimal length s.t. $\mu = w_\mu \cdot \mu^+$.

- Let $\mu$ be a composition such that $\mu_i < \mu_{i+1}$. Then

$$E_{s_i\mu} = t^{-1}\left(T_i + \frac{1-t}{1 - y_{i+1}(\mu)/y_i(\mu)}\right) E_\mu.$$

# Non-symmetric Macdonald Polynomials: Examples

- Example for $n = 2$.

$$
\begin{array}{lcl}
E_{(0,0)} & = & 1 \\
\hline
E_{(0,1)} & = & z_2 \\
\hline
E_{(1,0)} & = & z_1 + \left(\frac{1-t}{1-qt}\right) z_2 \\
\hline
E_{(1,1)} & = & z_1 z_2 \\
\hline
E_{(0,2)} & = & \left(\frac{q(1-t)}{1-qt}\right) z_1 z_2 + z_2^2 \\
\hline
E_{(2,0)} & = & z_1^2 + \left(\frac{q(1-t)^2}{(1-q^2t)(1-qt)} + \frac{1-t}{1-qt}\right) z_1 z_2 + \left(\frac{1-t}{1-q^2t}\right) z_2^2 \\
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$$

- $(0,0) \prec \boxed{(0,1) \prec (1,0)} \prec (1,1) \prec \boxed{(0,2) \prec (2,0)}$.
- The order is defined in [CGW20] [1].

$$\mu \geqslant \nu \Longleftrightarrow \sum_{i=1}^{j} \mu_i \geqslant \sum_{i=1}^{j} \nu_i, \quad \forall 1 \leqslant j \leqslant n.$$

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[1] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

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| $E_{(1,1)}$ | $=$ | $z_1 z_2$ |
| $E_{(0,2)}$ | $=$ | $\left(\frac{q(1-t)}{1-qt}\right) z_1 z_2 + z_2^2$ |
| $E_{(2,0)}$ | $=$ | $z_1^2 + \left(\frac{q(1-t)^2}{(1-q^2 t)(1-qt)} + \frac{1-t}{1-qt}\right) z_1 z_2 + \left(\frac{1-t}{1-q^2 t}\right) z_2^2$ |

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[1] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

# Non-symmetric Macdonald Polynomials: Examples

- Example for $n = 2$.

| | | | | | |
|---|---|---|---|---|---|
| $E_{(0,0)}$ | $=$ | $1$ | | | |
| $E_{(1,0)}$ | $=$ | $z_1$ | | | |
| $E_{(0,1)}$ | $=$ | $\left(\frac{1-t}{1-qt}\right) z_1$ | | $+z_2$ | |
| $E_{(1,1)}$ | $=$ | | | $z_1 z_2$ | |
| $E_{(2,0)}$ | $=$ | $z_1^2$ | $+\left(\frac{q(1-t)}{1-qt}\right)$ | $z_1 z_2$ | |
| $E_{(0,2)}$ | $=$ | $\left(\frac{1-t}{1-q^2 t}\right) z_1^2$ | $+\left(\frac{q(1-t)^2}{(1-q^2 t)(1-qt)} + \frac{1-t}{1-qt}\right)$ | $z_1 z_2$ | $+z_2^2$ |

- $(0,0) \prec \boxed{(1,0) \prec (0,1)} \prec (1,1) \prec \boxed{(2,0) \prec (0,2)}$.
- The order is defined in [HHL08] [1] also implemented in Sage.

$$\mu \geqslant \nu \iff \sum_{i=1}^{j} \mu_{n-i+1} \geqslant \sum_{i=1}^{j} \nu_{n-i+1}, \quad \forall 1 \leqslant j \leqslant n.$$

$$\mu \succ \nu \iff \mu^+ > \nu^+ \text{ or } \mu^+ = \nu^+, \mu > \nu.$$

[1] A combinatorial formula for nonsymmetric Macdonald polynomials (2008)

# Non-symmetric Macdonald Polynomials: Examples

**Conjecture 3.8** : Fix $\mu$ a composition. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$. If

$$\mathrm{Coeff}_p\left[E_\mu, m\right] := \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

exists and is non-zero. Then, there exists a unique $\nu$ for which

$$\mathrm{Coeff}_p\left[E_\mu, m\right] \propto E_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_\nu(z; q, t).$$

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- One can check some examples using Sage.
- Authors of [CGW20] [1] are able to show this conjecture only in some particular cases which is enough to construct some non-trivial duality functions.

[1] Integrable stochastic dualities and the deformed Knizhnik–Zamolodchikov equation (2020)

# Non-symmetric Macdonald Polynomials: Examples

**Prop. 3.5** : Define generating series

$$Y(w) = \sum_{i=1}^{n} Y_i w^i, \qquad y_\mu(w) = \sum_{i=1}^{n} y_i(\mu; q, t) w^i.$$

Then,
$$E_\mu(z; q, t) = \prod_{\nu \prec \mu} \frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot z^\mu.$$

- This can be seen as “Lagrange interpolation”.

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- This can be seen as "Lagrange interpolation".

Proof: Use the following two properties:

- $\frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot E_\nu = \begin{cases} 0, & \text{if } \nu \prec \mu, \\ E_\mu, & \text{if } \nu = \mu. \end{cases}$
- $E_\mu = z^\mu + \sum_{\nu \prec \mu} c_{\mu,\nu}(q, t) z^\nu.$

□

# Non-symmetric Macdonald Polynomials: Coefficients

**Prop. 3.6** : Fix $\mu$ a composition. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$. If

$$\mathrm{Coeff}_p\,[E_\mu, m] := \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

exists and is non-zero. Then,

$$\mathrm{Coeff}_p\,[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p \sum_{\nu \in \mathcal{E}_\mu} c_\nu(q, t) E_\nu(z; q, t),$$

where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$.

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where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$.

Recall that eigenvalues are $y_i(\mu; q, t) = q^{\mu_i} t^{\rho(\mu)_i + n - i + 1}$.

Thus,
$$\begin{aligned} y_\mu(w) = y_\nu(w) &\iff y_i(\mu) = y_i(\nu), && \forall 1 \leqslant i \leqslant n, \\ &\iff q^{\mu_i} t^{\rho(\mu)_i} = q^{\nu_i} t^{\rho(\nu)_i}, && \forall 1 \leqslant i \leqslant n. \end{aligned}$$

Unique solution $\nu = \mu$ for generic $q$ and $t$; more solutions at $q = t^{-m}$.

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where $\mathcal{E}_\mu = \{\nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m}\}$.

Proof: Use "Lagrange interpolation",

$$\prod_{\substack{\nu \prec \mu \\ \nu \notin \mathcal{E}_\nu}} \frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot z^\mu = E_\mu + \sum_{\nu \in \mathcal{E}_\mu} d_{\mu,\nu}(q, t) E_\nu.$$

By taking $\lim_{q \to t^{-m}} (1 - qt^m)^p$ on both sides, LHS = 0 and RHS gives the proposition. □

# Non-symmetric Macdonald Polynomials: Coefficients

**Prop. 3.7** : Assume that $p = |\mathcal{E}_\mu|$, then there exists a unique $\nu$ for which

$$E_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_\nu(z; q, t)$$

is well-defined and $\mathrm{Coeff}_p\,[E_\mu, m] \propto E_\nu(z; t^{-m}, t)$.

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- Conjecture 3.8 is true with an additional condition $p = |\mathcal{E}_\mu|$.
- Proof uses "Lagrange interpolation" and by induction.
- The composition $\nu$ corresponds to the minimal composition in $\mathcal{E}_\mu$.

## Outline

- The notion of the duality: operator / matrix formalism.
- Multi-species ASEP process.
- Hecke algebra, tKZ equation and trivial duality function.
- Non-symmetric Macdonald polynomials.
- ASEP polynomials: construction of non-trivial duality functions.
- Matrix product Ansatz and examples.

# ASEP polynomials: Definition

- Another basis ASEP polynomials can be defined from NSMP,

$$f_\delta(z;q,t) = E_\delta(z;q,t), \qquad \forall \delta = (\delta_1 \leqslant \ldots \leqslant \delta_n),$$
$$f_{s_i\mu} = T_i^{-1} f_\mu, \qquad \mu_i < \mu_{i+1}.$$

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> Recall that $E_{s_i\mu} = t^{-1}\left(T_i + \frac{1-t}{1-y_{i+1}(\mu)/y_i(\mu)}\right)E_\mu$ for $\mu_i < \mu_{i+1}$.

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and satisfies (tKZ).
- $$f_{\mu_n,\mu_1,\ldots,\mu_{n-1}}(qz_n, z_1, \ldots, z_{n-1}; q, t) = q^{\mu_n} f_\mu(z;q,t). \qquad \text{(Cyclic-B.C.)}$$

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- Above two properties uniquely define ASEP polynomials.
- ASEP and NSMP are also related by a triangular change of basis.

# ASEP polynomials: Definition

**Prop. 3.10** : For any composition $\mu$, the following expansions are unique,

$$E_\mu(z;q,t) = f_\mu(z;q,t) + \sum_{\substack{\nu\in\sigma(\mu)\\ \nu\prec\mu}} c_{\mu,\nu}(q,t) f_\nu(z;q,t);$$

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Proof: For an anti-partition, $E_\delta = f_\delta$ so the proposition is true. By induction, assume $\mu$ is s.t. the proposition holds with $\mu_i < \mu_{i+1}$,

$$E_{s_i\mu} = t^{-1}\left(T_i + \frac{1-t}{1-y_{i+1}(\mu)/y_i(\mu)}\right)\left(f_\mu + \sum_{\substack{\nu\in\sigma(\mu)\\ \nu\prec\mu}} c_{\mu,\nu}(q,t) f_\nu\right).$$

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- $T_i f_\nu = (t-1)f_\nu + t f_{s_i\nu}$;
- $\nu, s_i\nu \in \sigma(\mu) = \sigma(s_i\mu)$;
- $(\nu \prec \mu, \mu \prec s_i\mu) \Rightarrow \nu, s_i\nu \prec s_i\mu$.

## ASEP polynomials: Definition

**Theorem 3.11** : Fix an anti-partition $\delta$. Let $m \in \mathbb{Q}_{>0}$ and $p \in \mathbb{N}$ such that Conjecture 3.8 holds. Then, there exists a unique anti-partition $\varepsilon$

$$f_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} f_\nu(z; q, t)$$

is well-defined for all $\nu \in \sigma(\varepsilon)$ and

$$\mathrm{Coeff}_p\,[f_\mu, m] = \sum_{\nu \in \sigma(\varepsilon)} \Psi(\nu, \mu; t) f_\nu(z; t^{-m}, t)$$

for all $\mu \in \sigma(\delta)$ and suitable coefficients $\Psi(\nu, \mu; t)$.

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for all $\mu \in \sigma(\delta)$ and suitable coefficients $\Psi(\nu, \mu; t)$.

Proof:

- The proof is based on Prop. 3.10.
- Use the recurrence relation and the fact that $T_i$ commutes with **lim**.

# ASEP polynomials: Definition

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## ASEP polynomials: Definition

**Theorem 3.12** : Keep the notations from Theorem 3.11. $\Psi(\nu, \mu; t)$ defines a local duality function.

Proof: From Prop 3.2 (trivial duality function), the following holds

$$\mathbb{L}_i|\mathcal{I}\rangle = \mathbb{M}_i|\mathcal{I}\rangle, \quad 1 \leqslant i \leqslant n-1, \tag{1}$$

where $|\mathcal{I}\rangle = \sum_{\mu} f_\mu(z; q, t)|\mu\rangle$. Taking the coefficient,

$$|\mathcal{I}_{p,m}\rangle := \mathrm{Coeff}_p\left[|\mathcal{I}\rangle, m\right] = \sum_{\mu\in\sigma(\delta)}\sum_{\nu\in\sigma(\varepsilon)} \Psi(\nu, \mu; t) f_\nu(z; t^{-m}, t)|\mu\rangle$$

also satisfies $\mathbb{L}_i|\mathcal{I}_{p,m}\rangle = \mathbb{M}_i|\mathcal{I}_{p,m}\rangle, \quad 1 \leqslant i \leqslant n-1.$

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# Matrix product Ansatz

- The coefficients $\Psi(\nu, \mu)$ are difficult to compute in general.
- The matrix product Ansatz from [CGW15] [1] is useful for ASEP polynomials.

**Theorem** : Fix $r \geqslant 1$. For a composition $\mu$ with largest part equal to $r$,

$$f_\mu(z; q, t) = \Omega_\mu(q, t) \times \mathrm{Tr}\Big(A_{\mu_1}(z_1) \dots A_{\mu_n}(z_n) S\Big), \qquad \text{(Mat-Ans)}$$

where $\{A_i(z)\}_{0 \leqslant i \leqslant r}$ and $S$ are explicit matrices.

Proof: Translate (tKZ) and (Cyclic-B.C.) into matrices.

[1] Matrix product formula for Macdonald polynomials (2015)

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