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The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems

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A systematic method is developed which allows one to identify certain important classes of evolution equations which can be solved by the method of inverse scattering. The form of each evolution equation is characterized by the dispersion relation of its associated linearized version and an integro-differential operator. A comprehensive presentation of the inverse scattering method is given and general features of the solution are discussed. The relationship of the scattering theory and Backlund transformations is brought out. In view of the role of the dispersion relation, the comparatively simple asymptotic states, and the similarity of the method itself to Fourier transforms, this theory can be considered a natural extension of Fourier analysis to nonlinear problems.

I. Introduction

Many of the hardest problems and most interesting phenomena being studied by mathematicians, engineers and physicists are nonlinear in nature. Often, these phenomena can be modeled (and there is good reason to believe that these models are accurate) by nonlinear partial differential equations and to be sure, it will be many years to come before we have the mathematical sophistication to handle these equations completely. To this date, the successful analysis has usually depended on the ability to decompose the partial differential equations into a set of ordinary differential equations. For example, in systems which are essentially hyperbolic, this decomposition is achieved by transformations which introduce characteristic coordinates and much progress has resulted in our understanding of certain propagation phenomena associated with nondispersive waves. In most cases, however, the analyst is reduced either to finding special solutions (which are in many cases both stable and realizable solutions) or essentially to linearizing the system and exploiting the smallness of some parameter. While the latter approach is often effective in getting the investigation off the ground so that one can study weak interactions, many of the problems mentioned above simply cannot be considered as perturbations around linear states and the act of linearization merely serves to remove from the equations their most interesting properties.

Nevertheless, it is worth dwelling on the reason for the analyst's success in dealing with linear problems. The basic reason for this success is that he can take advantage of the superposition principle and construct any solution of the equation system as a linear combination of the normal modes of that system. The analysis consists of first finding the normal modes and then finding the coefficients involved in their linear combination which satisfies the specified initial and boundary data.

The purpose of this paper is to show that a certain class of nonlinear partial differential equations can be handled in an analogous fashion by a technique which we call the inverse scattering transform (IST). In all cases we will be dealing with evolution equations, by which we mean that the equation describes how a particular quantity evolves in time from a specified initial state. The method was first discovered by Gardner, Greene, Kruskal and Miura (GGKM, 1968) and applied by them to the Korteweg deVries (KdV, 1895) equation in a truly pioneering work. It has many features in common with the method of Fourier transforms and may be considered an extension of Fourier analysis to nonlinear problems.

It is useful to recall some of the essential features of the Fourier transform method as applied to the linear equation

$$u_t(x,0) = -i\omega\left(-i\frac{\partial}{\partial x}\right)u(x,t) \qquad -\infty < x < \infty \tag{1.1}$$

where $u(x, 0)$ is specified and $\omega(k)$ is the dispersion relation. The unknown variable $u(x, t)$ can be mapped into its Fourier transform $\hat{u}(k, t)$ by

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx \qquad -\infty < k < \infty \tag{1.2}$$

and in particular $\hat{u}(k, 0)$ may be computed. The time evolution of $\hat{u}(k, t)$ for each k can be traced separately,

$$\hat{u}_t(k,t) = -i\omega(k)\hat{u}(k,t). \tag{1.3}$$

Thus, given $u(x, 0)$ one can find $\hat{u}(k, 0)$ with (1.2), trace its time evolution with (1.3) and finally recover the unknown quantity $u(x, t)$ from the inverse mapping:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{ikx}\, dk. \tag{1.4}$$

Two points are worth noting. Firstly, it is the separability (the time evolution of $\hat{u}(k, t)$ is only determined by functionals evaluated at k) of (1.1) which is the key to obtaining global (in time) solutions. Secondly, the spectrum (i.e. the set of wavenumbers required to represent the solution at any time), is time invariant.

Let us now turn our attention to nonlinear problems and consider for purposes of illustration the Korteweg deVries equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{1.5}$$

when the initial data $u(x, 0)$ is given and $u(x, t) \to 0$ sufficiently rapidly as $|x| \to \infty$. Associate with (1.5) the linear eigenvalue problem,

$$v_{xx} + (\zeta^2 + \tfrac{1}{6}u(x,t))v = 0 \qquad -\infty < x < \infty \tag{1.6}$$

which is the Schroedinger equation with $u(x, t)$ playing the role of the potential. The eigenvalues of (1.6) may be computed and play the analogous role to the spectrum of a linear system. For $u(x, t)$ real, the eigenvalues are the set of all real ζ (the continuous spectrum) and a finite number of distinct imaginary numbers $\zeta = iK_n$, $n = 1, \ldots, N$ (the discrete spectrum). The eigenfunctions corresponding to these eigenvalues may be computed and in particular their asymptotic behavior may be written: for $\zeta = k$, k real,

$$\begin{aligned} v(k, x, t) &\to e^{-ikx} + R(k, t)\, e^{ikx}, && x \to +\infty \\ &\to T(k, t)\, e^{-ikx}, && x \to -\infty, \end{aligned} \tag{1.7a}$$

which relations serve to define the reflection coefficient $R(k, t)$ and the transmission coefficient $T(k, t)$; for $\zeta = iK_n$,

$$\begin{aligned} v_n(x, t) &\to C_n(t)\, e^{-K_n x}, && x \to +\infty \\ &\to D_n(t)\, e^{K_n x}, && x \to -\infty \end{aligned} \tag{1.7b}$$

and for convenience v_n is normalized by $\int_{-\infty}^{\infty} |v_n|^2\, dx = 1$.

Essentially, then, the association of the solution of (1.5) with the scattering problem (1.6) gives a mapping between $u(x, t)$ and what we shall call the scattering data S

$$u(x, t) \to S(\{iK_n, C_n(t)\}_{n=1}^{N}, T(k, t), R(k, t)) \tag{1.8}$$

which is a composition of the spectrum ($\{iK_n\}_{n=1}^{N}$, $-\infty < k < \infty$) and the coefficients representing the asymptotic behavior of the corresponding eigenfunctions. So far in the discussion the time parameter t has played no role. Indeed since only $u(x, 0)$ is known one can only find the scattering data S at the initial time. However, by analogy with the linear problem one may attempt to find the time evolution of S. GGKM [1967] showed that as $u(x, t)$ evolves according to (1.5), the discrete eigenvalues $\zeta = iK_n$ remain time invariant and the eigenfunctions $v(\zeta, x, t)$ evolve according to

$$v_t = \left(\frac{u_x}{6} + c\right) v + \left(4\zeta^2 - \frac{u}{3}\right) v_x, \tag{1.9}$$

where $c = 4iR^3$, ζ real and $c = 0$, ζ imaginary. The reader may verify by cross differentiation that (1.6) and (1.9) guarantee that $(\zeta^2)_t = 0$ if $u(x, t)$ satisfies (1.5). While the determination of $v(\zeta, x, t)$ at later times depends on a knowledge of $u(x, t)$, the asymptotic behavior of $v(\zeta, x \to \pm\infty, t)$ does not! This is one of the crucial features in the success of the inverse scattering transform. Therefore, the time evolution of the scattering data S can be found and is given by remarkably simple expressions in which the time behavior of the scattering data is again separable:

$$\begin{aligned} (K_n)_t &= 0, \qquad n = 1, \ldots, N \\ T_t(k, t) &= 0 \\ R_t(k, t) &= 8ik^3 R(k, t) \\ (C_n^2)_t(t) &= 8K_n^3 C_n^2(t). \end{aligned} \tag{1.10}$$

It is no coincidence that the time evolution of the reflection coefficient is directly related to the dispersion relation of the linearized KdV equation, a point we shall discuss later.

Given the scattering data at the initial time one can compute its time evolution from (1.10). It is now very natural to ask if the mapping (1.8) can be inverted. This question was answered affirmatively by Gelfand and Levitan [1951] in a classic paper well before the question arose in its present context. The inverse process may be specified. Define

$$B(p, t) = \sum_{n=1}^{N} C_n^2(t)\, e^{-K_n p} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k, t)\, e^{ikp}\, dk \tag{1.11}$$

and solve the linear integral equation

$$K(x, y, t) + B(x + y, t) + \int_{x}^{\infty} K(x, z, t)B(y + z, t)\, dz = 0 \tag{1.12}$$

in the region $y > x$, subject to the boundary condition

$$K(x, z) \to 0, z \to +\infty.$$

Then, the potential $u(x, t)$ is given by

$$u(x, t) = 12\frac{d}{dx}K(x, x; t). \tag{1.13}$$

The method is therefore analogous to the Fourier transform method of linear problems: namely, one maps the initial data into the scattering data, follows the evolution of the set of scattering data and at any desired time inverts the mapping with (1.12), thereby recovering the solution $u(x, t)$ to the partial differential equation (1.5). We may summarize the situation schematically as follows. Linear problems:

$$u(x, 0) \to \hat{u}(k, 0) \xrightarrow{\omega(k)} \hat{u}(k, t) \to u(x, t).$$

Nonlinear problems:

$$\begin{aligned} u(x, 0) &\to S(\{iK_n, C_n(0)\}_{n=1}^{N}, T(k, 0), R(k, 0)) \\ &\xrightarrow{\omega(2k)} S(\{iK_n, C_n(t)\}_{n=1}^{N}, T(k, t), R(k, t)) \\ &\to u(x, t). \end{aligned}$$

Two features, mentioned before, are well worth stressing again. Firstly, the distinctly new feature of the nonlinear problem is the occurrence of some "normal modes" (corresponding to the discrete eigenvalues of (1.6)) whose values depend on the initial data. These eigenvalues give rise to solutions of (1.5) known as solitons which are distinctly nonlinear entities. These solutions arise as a balance between the nonlinear and dispersive term in (1.5) and cannot be obtained as a nonlinear extension of any linear solution mode. Secondly, the success of the method depends critically on the facts that (a) only the scattering data (i.e., the asymptotic behavior of the eigenfunctions) is required to determine the potential $u(x, t)$ uniquely, (b) the time evolution of the asymptotic behavior of the eigenfunctions is independent of the unknown quantity $u(x, t)$. The structure of the normal modes (the eigenfunctions) for all x is not required. If it were, the power of the method would be lost.

When the method was first presented as a means to solve the KdV equation, there was much speculation as to whether the method was a fluke. However, in 1972, Zakharov and Shabat [1972] used the scattering problem

$$\begin{aligned} v_{1x} + i\zeta v_1 &= q(x,t)v_2 \\ v_{2x} - i\zeta v_2 &= r(x,t)v_1 \end{aligned} \qquad -\infty < x < \infty \tag{1.14}$$

with $r = -q^*$, to find the initial value solution for the nonlinear Schroedinger equation

$$q_t - iq_{xx} - 2iq^2q^* = 0. \tag{1.15}$$

This second successful implementation of IST was strong evidence that the method was not fortuitous and indeed applicable to a broad class of evolution equations.

We intend to present in this paper an extension of these ideas and will show that the method can be used to solve the coupled equations

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} + 2A_0(L^+)\begin{pmatrix} r \\ q \end{pmatrix} = 0 \tag{1.16}$$

where L^+ is the integro-differential operator

$$L^+ = \frac{1}{2i}\begin{pmatrix} \dfrac{\partial}{\partial x} - 2r\displaystyle\int_{-\infty}^{x} dy\, q & 2r\displaystyle\int_{-\infty}^{x} dy\, r \\ -2q\displaystyle\int_{-\infty}^{x} dy\, q & -\dfrac{\partial}{\partial x} + 2q\displaystyle\int_{-\infty}^{x} dy\, r \end{pmatrix} \tag{1.17}$$

and $A_0(\zeta)$ is an arbitrary ratio of entire functions. Indeed, it is clear that $A_0(\zeta)$ is directly related to the dispersion relation of the linearized version of (1.16):

$$A_0(\zeta) = \frac{i}{2}\omega_r(2\zeta) = -\frac{i}{2}\omega_q(-2\zeta),$$

where ω_r and ω_q are the linearized dispersion relations for the r and q equations, respectively. A central point of this paper is: given a dispersion relation $\omega(k)$ mereomorphic, and real for real k, there is a nonlinear evolution equation whose linearized version has this dispersion relation and for which appropriate initial value problems can be solved exactly. The underlying scattering problem for this class of equations is (1.14). Included are such well-known equations as the sine-Gordon, sinh-Gordon, nonlinear Schroedinger (stable and unstable cases), and the modified Korteweg deVries [Ablowitz, Kaup, Newell and Segur, 1973b]. Also included are all linear equations, for which cases the method reduces essentially to the Fourier transform method. The reader will note that when $r = -q^*$ and $A_0(\zeta) = -2i\zeta^2$, (1.16) is (1.15). When the expression $A_0(\zeta)$ has poles, some care must be exercised in interpreting the evolution equations obtained. It turns out to be convenient in these cases to write the evolution equation as a system in which quadratic products of the eigenfunction components v_1, v_2 found in (1.14) play the role of auxiliary dependent variables. This fact suggests that the eigenvalue problem (1.14) may have physical significance in these contexts.

We can also find the exact solution for the set of problems for which the Schroedinger equation,

$$v_{xx} + (\zeta^2 + q(x,t))v = 0 \qquad -\infty < x < \infty,$$

is the appropriate scattering problem. This class is given by

$$q_t + C_0(L_s^+)q_x = 0, \tag{1.18}$$

where

$$L_s^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}q_x\int_x^\infty dy, \tag{1.19}$$

and $C_0(k^2)$ is an arbitrary ratio of entire functions, directly related to the phase speed in the linearized problem.

Many of the evolution equations contained in the classes (1.16), (1.18) have wide application and arise as the limiting equations of more complicated systems. For example, the KdV equation arises in all systems [see for example: Gardner and Morikawa, 1960; Zabusky, 1967; Kruskal, 1963; Washimi and Taniuti, 1966; Wijngaarden, 1968; Leibovich, 1970; Nariboli, 1969; and Benney, 1974] which are weakly dispersive and weakly nonlinear. The nonlinear Schroedinger equation [Talanov, 1967; Kelley, 1965; Bespalov, Litvak and Talanov, 1968; and Benney and Newell, 1967] arises as the envelope equation of a dispersive wave system which is almost monochromatic and weakly nonlinear. The sine-Gordon equation [Frenkel and Kontorova, 1939; Enz, 1963; Skyrme, 1958; Skyrme, 1961; Scott, 1967; and Ablowitz, Kaup, Newell and Segur; 1973a] also arises in many physical contexts but in situations which cannot be so readily characterized. An excellent account of the initial development of the method, the origin of these equations and the properties of solutions is given by Scott, Chu and McLaughlin 1973.

While many of the evolution equations which fall into the classes given by (1.16) and (1.18) arise in vastly different contexts, there are distinct features common to all the solutions. The first is the concept of the soliton (a term first coined by Zabusky and Kruskal (1965) which is a stable, localized, permanent waveform which evolves in time by simple translation, a characteristic first noted and described by Russell as long ago as 1844. These entities are also known as kinks (in the context of sine-Gordon) and 2π pulses (in the context of the self-induced transparency phenomenon) and each corresponds to one of the time invariant discrete eigenvalues of the scattering problem. The soliton displays the well-known character of nonlinear solutions in that its amplitude, velocity and shape are all interrelated and given in terms of the discrete eigenvalue ζ. Its relative position is determined by the amplitude of the corresponding eigenfunction at time zero. Because of the invariance of the eigenvalues, the soliton displays remarkable interaction properties with other solitons of different velocities. In fact, while the interaction itself is extremely complicated and nonlinear, each soliton eventually emerges unscathed, retaining all components of its original character except for a phase shift. Because the soliton speed is a function of ζ, there is, in general, a locus in the complex ζ plane on which this speed is constant. If the initial data is such that two or more of the bound states lie on this locus then a multisoliton bound structure is formed. A particular example is given by the solution which we have termed the breather [Ablowitz, Kaup, Newell and Segur, 1973a] (so called because it pulsates as it travels) which is a multisoliton structure formed by the discrete eigenvalues ζ and $-\zeta^*$. These solutions have been termed 0π solutions by Lamb [1971] because their integral over space at fixed time is zero.

We will discuss and list these solutions in detail in Section V. In most examples to date, the manifestation of the discrete eigenvalue in physical space has been the extremely stable and well-behaved entity to which we have referred as a soliton. However, there are situations in which the solution corresponding to a discrete eigenvalue displays rather unexpected properties.

A second distinct feature of the general solution is a radiation component which in general is not localized, does not have a permanent wave form and, as it spreads out, decays algebraically in time. While these two solution components are separable in the transform space, (the solitons are identified with the discrete eigenvalues; the radiation is identified with the continuous spectrum), it is difficult to separate these components in physical space. However, when the solutions are valid for long time, the radiation has significantly decayed and the solitons can be seen as localized pulses with permanent shape traveling in a sea of radiation. In some cases, like the KdV equation, the solitons and the radiation become completely separated with the solitons moving to the right ($x > 0$) and the radiation to the left ($x < 0$). In this situation, a third solution feature arises. This feature is the similarity solution originally found by Berezin and Karpman [1964] which provides the structure in the intervening region between the solitons and the radiation.

The detailed structure of the radiation field is difficult to calculate for all time but it can be examined in the long time limit by the traditional asymptotic methods [see Ablowitz and Newell, 1973]. Although nonlinear, it is that part of the solution to the nonlinear problem which most closely resembles the solution to linear problems. Indeed, in the case of the KdV equation, the radiation component can be computed by successively iterating the linear solution [Segur, 1973]. Thus, the basic oscillatory structure of the radiation has a linear character and exhibits the usual decay property associated with linear dispersive waves. In Section V, we give a specific criterion as to when the problem may be treated by a perturbation approach about some linear state.

In causal problems, such as the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{1.20}$$

the continuous spectrum plays a very important role in that it ensures that the solution remains causal. If the initial data is on compact support, the solitons and the radiation field together ensure that the solution remains on compact support at any future time. Even in noncausal problems such as the KdV equation, there is a similar effect.

Some major questions still remain, the answers to which are only partially understood. Why is the Schroedinger equation the appropriate linear eigenvalue problem for the KdV equation and the class of equations (1.18)? Why is (1.14) suitable for the class of evolution equations (1.16)? Given a specific evolution equation, can one find a linear eigenvalue problem by which the given equation may be solved? What particular properties characterize these problems which are solvable by the inverse scattering transform method?

One of these properties is the existence of an infinite sequence of conservation laws of the form

$$\frac{\partial T_n}{\partial t} + \frac{\partial F_n}{\partial x} = 0, \quad n = 1, 2, 3, \ldots$$

and in most (but not all) cases a corresponding sequence of motion invariants given by the integrated densities $\int T_n \, dx$. Historically the conservation laws have played an important role in the development of the inverse scattering transform. Indeed it was a transformation between solution of the KdV and modified KdV equations which led GGKM to the choice of the Schroedinger equation. Miura [1968] was led to this transformation by attempting to match the corresponding sets of conservation laws for the two equations. This transformation is a special case of a class of transformations (called Backlund transformations) which seem to be destined to play an important role in the development of the theory. Indeed, we will show that, associated with each evolution equation compatible with (1.14), there is a unique system of equations describing the time evolution of the eigenfunctions defined by (1.14). This dual system can be directly converted into a Backlund transformation relating solutions of the given evolution equation. The process can also be reversed. One entertains the hope, therefore, that an independent means of constructing the Backlund transformations can be found and thereby provide a constructive means of choosing, for a given evolution equation, the appropriate linear eigenvalue problem. They also provide the analogous "superposition" principle for this class of nonlinear equations.

As a summary to this discussion, we outline the material which we cover in the various sections. The principle aim of the paper is to find the broadest class of evolution equations which are associated with (1.14) and (1.6) and which are solvable by the inverse scattering transform. In Sections II and III (some of the material in these sections is amplified in Appendices 1–4), we follow a sequence of steps in which each further generalization arises as a more or less natural sequel to the proceeding step. We not only arrive at a stage where we can show how equations of the form (1.16) and (1.18) can be solved, but we also indicate how to proceed beyond this class. In Section IV, (some of this material is amplified in Appendices 5–7), we present a detailed discussion of the direct and inverse scattering problem associated with (1.14) and spell out the mathematical requirements of the dependent variables $q(x, t)$ and $r(x, t)$. In Section V, we discuss the solutions of equations of the form (1.18). Section VI provides approximation methods to estimate the discrete spectrum. In Section VII, we discuss the role of Backlund transformations and the existence and construction of the conservation laws and the conditions under which motion invariants occur. We also mention how the Hopf–Cole transformation and its generalization are analogous to IST.

II. Relation of eigenvalue problem to evolution equations

We consider the eigenvalue problem,

$$v_{1x} + i\zeta v_1 = q(x, t)v_2 \tag{2.1a}$$

$$v_{2x} - i\zeta v_2 = r(x, t)v_1 \tag{2.1b}$$

on the interval $-\infty < x < \infty$, where the parameter ζ plays the role of the eigenvalue and the "potentials" $q(x, t)$ and $r(x, t)$ are the solutions of a coupled pair of nonlinear evolution equations to be discussed below. We assume $q(x, 0)$ and $r(x, 0)$ are given and have the prerequisite properties for the analysis to be valid. These properties are spelled out in detail in Section IV. The parameter t corresponds to time and we expect that, in general, the eigenfunctions and eigenvalues of (2.1) will evolve in time as the potentials $q(x, t)$ and $r(x, t)$ evolve according to some evolution equation. Accordingly, we choose the time dependence of $v_1(x, t)$ and $v_2(x, t)$

$$v_{1t} = A(x, t, \zeta)v_1 + B(x, t, \zeta)v_2 \tag{2.2a}$$

$$v_{2t} = C(x, t, \zeta)v_1 + D(x, t, \zeta)v_2. \tag{2.2b}$$

We now insist that the eigenvalue be independent of time. From cross differentiation of (2.1) and (2.2) it is found that

$$D = -A + d(t)$$

(without loss of generality we set $d(t) \equiv 0$) and the following equations [Ablowitz, Kaup, Newell and Segur, 1973b] for $A(x, t, \zeta)$, $B(x, t, \zeta)$, $C(x, t, \zeta)$,

$$A_x = qC - rB \tag{2.3a}$$

$$B_x + 2i\zeta B = q_t - 2Aq \tag{2.3b}$$

$$C_x - 2i\zeta C = r_t + 2Ar. \tag{2.3c}$$

The equations (2.1), (2.2) and (2.3) form the basis of all the relevant mathematical analysis. Equation (2.1) is used to find the discrete eigenvalues (which are time invariant) and the asymptotic behavior ($|x| \to \infty$) of all the eigenfunctions $v_1(x, 0, \zeta)$, $v_2(x, 0, \zeta)$ at the initial time. Given a specific pair of evolution equations (namely, specifying q_t and r_t) we may in principle compute solutions to equation (2.3). Provided these solutions have reasonable properties (and they will only have these for a restricted class of q_t and r_t), we can compute from (2.2), the time evolution of the asymptotic behavior ($|x| \to \infty$) of the eigenfunctions v_1 and v_2. This information is sufficient to enable us to reconstruct the potentials $q(x, t)$ and $r(x, t)$ at later times.

Our strategy will not be to specify q_t and r_t a priori, but rather to find the solutions to (2.3) for a general q_t and r_t. As we have indicated only a certain subclass of the solutions to (2.3) allow simple expressions for the time evolution of the eigenfunctions of (2.1). We then determine, a posteriori, the class of evolution equations which correspond to this subclass of solutions to (2.3).

As a first step, we will find some very simple solutions to (2.3), which although rather special, yield a very broad class of evolutions equations. If we set

$$A = \sum_0^N A^{(n)}\zeta^n, \qquad B = \sum_0^N B^{(n)}\zeta^n, \qquad C = \sum_0^N C^{(n)}\zeta^n, \tag{2.4}$$

we find $A^{(N)} = a_N$ (independent of x; can depend on t), $B^{(N)} = C^{(N)} = 0$. We use the last two equations in (2.3) to solve for $B^{(N-1)}$, $C^{(N-1)}$ and the first to solve for $A^{(N-1)}$. Repeating the process gives all the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$. In particular the evolution equations are determined by the ζ^0 balance by the last two equations in (2.3), namely,

$$q_t = 2A^{(0)}q + B_x^{(0)}$$
$$r_t = -2A^{(0)}r + C_x^{(0)}. \tag{2.5}$$

To illustrate, let us take $N = 3$. Then, we find,

$$\begin{aligned} A &= a_3\zeta^3 + a_2\zeta^2 + \left(\frac{1}{2}a_3qr + a_1\right)\zeta + \frac{1}{2}a_2qr - \frac{i}{4}a_3(qr_x - q_xr) + a_0, \\ B &= ia_3q\zeta^2 + \left(ia_2q - \frac{1}{2}a_3q_x\right)\zeta + \left(ia_1q + \frac{i}{2}a_3q^2r - \frac{1}{2}a_2q_x - \frac{i}{4}a_3q_{xx}\right), \\ C &= ia_3r\zeta^2 + \left(ia_2r + \frac{1}{2}a_3r_x\right)\zeta + \left(ia_1r + \frac{i}{2}a_3qr^2 + \frac{1}{2}a_2r_x - \frac{i}{4}a_3r_{xx}\right) \end{aligned} \tag{2.6}$$

together with the evolution equations

$$\begin{aligned} 0 &= q_t + \frac{i}{4}a_3(q_{xxx} - 6qrq_x) + \frac{1}{2}a_2(q_{xx} - 2q^2r) - iaq_x - 2a_0q \\ 0 &= r_t + \frac{i}{4}a_3(r_{xxx} - 6qrr_x) - \frac{1}{2}a_2(r_{xx} - 2qr^2) - ia_1r_x + 2a_0r. \end{aligned} \tag{2.7}$$

As special cases, we list,

(i) $a_0 = a_1 = a_2 = 0, a_3 = -4i$

(a) $r = -1, q_t + 6qq_x + q_{xxx} = 0$ (KdV) (2.8)

(b) $r = \mp q, q_t \pm 6q^2q_x + q_{xxx} = 0$ (modified KdV), (2.9)

(ii) $a_0 = a_1 = a_3 = 0, a_2 = -2i$

(a) $r = \mp q^*, q_t - iq_{xx} \mp 2iq^2q^* = 0$ (nonlinear Schroedinger). (2.10)

Equation (2.8) is the well-known Korteweg-deVries equation. When $r = -1$, the scattering problem (2.1) reduces to the Schroedinger equation

$$v_{2xx} + (\zeta^2 + q(x,t))v_2 = 0. \tag{2.11}$$

We know for real $q(x, t)$, ζ^2 is real and the discrete eigenvalues, each of which corresponds to stable localized wave pulse known as a soliton, all lie on the imaginary ζ axis. In general the discrete eigenvalues, corresponding to the localized pulses in the solution $q(x, t)$ are complex. However, in the case where the operator (note (2.1) is $\mathscr{L}v = \zeta v$)

$$\mathscr{L} = \begin{pmatrix} i\dfrac{\partial}{\partial x} & -iq \\ ir & -i\dfrac{\partial}{\partial x} \end{pmatrix}$$

is equal to its Hermitian adjoint, ζ is real and there are no bound states (proper eigenvalues). This occurs only when $r = q^*$ and corresponds to the lower signs on the nonlinear terms in equations (2.9) and (2.10).

In the same way that we found the evolution equations corresponding to A, B, C being expanded in a positive power series around $\zeta = 0$, we can also find equations for which A, B, C are given in inverse powers. For example, if we take,

$$A(x, t, \zeta) = \frac{a(x, t)}{\zeta}, \qquad B(x, t, \zeta) = \frac{b(x, t)}{\zeta}, \qquad C(x, t, \zeta) = \frac{c(x, t)}{\zeta} \tag{2.12}$$

we find

$$a_x = \frac{i}{2}(qr)_t$$

$$q_{xt} = -4iaq, \qquad r_{xt} = -4iar. \tag{2.13}$$

As special but important cases, we list

(i) $a = \frac{i}{4}\cos u,\ b = c = \frac{i}{4}\sin u,\ r = -q = \frac{1}{2}u_x,$

$$u_{xt} = \sin u \quad \text{(sine-Gordon equation)} \tag{2.14}$$

and

(ii) $a = \frac{i}{4}\cosh u,\ -b = c = \frac{i}{4}\sinh u,\ r = q = \frac{1}{2}u_x,$

$$u_{xt} = \sinh u \quad \text{(sinh-Gordon equation).} \tag{2.15}$$

We have listed here a few of the better known evolution equations which can be solved by the inverse scattering method. Clearly many more can be generated and in Section III we demonstrate how a broader class of equations can be derived and how each is related to its linearized dispersion relation.

III. General evolution equations

An examination of the results of Section II suggests the following question: Are the only evolution equations solvable by this scattering problem the result of finite power series expansions in ζ? In this section we show that a wider class of equations exists. It turns out that:

(i) the system (2.1) and (2.2) with appropriate side conditions, completely determine the evolution equation for which the inverse transform applies;

(ii) the general evolution equation is characterized by a simply calculated function (related to the dispersion relation) and an integro-differential operator;

(iii) the technique extends beyond those of previous workers (Lax, 1968, Zakharov and Faddeev, 1971) to nonpolynomial dispersion relations, and can result in operator equations!

The key step in ascertaining the general evolution operator is to work closely with the equations governing A, B, C (2.3). A crucial point to note, however, is that the solution A, B, C given in Section II all obey the boundary conditions:

$$A(x, t, \zeta) \to A_0(\zeta)$$

$$B(x, t, \zeta) \to 0$$

$$C(x, t, \zeta) \to 0 \qquad \text{as } |x| \to \infty.$$

Clearly, then, the only way for the system (2.3) to have a solution is if the inhomogeneous terms (q_t, r_t) satisfy certain orthogonality conditions. These conditions yield (1.16)!

In the more general case A, B, C take on different values on the left $(x \to -\infty)$ and right $(x \to +\infty)$ and certain integral relations can also be extracted. It will be shown in a future paper that the self-induced transparency (S.I.T.) equations of nonlinear optics [Lamb, 1973] are a special case of these more general boundary conditions.

In order to deduce the necessary integral conditions we shall formally solve the system (2.3). This solution is easily given in terms of specific solutions of (2.1). Hence, let us first examine briefly the fundamental solutions of this eigenvalue problem.

Assuming $q(x, t)$ and $r(x, t) \to 0$ as $|x| \to \infty$, define the linearly independent solutions to (2.1) which (for ζ real), have the following asymptotic values:

$$\phi \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x} \qquad x \to -\infty$$

$$\bar{\phi} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\zeta x} \qquad x \to -\infty \tag{3.1}$$

$$\psi \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x} \qquad x \to +\infty$$

$$\bar{\psi} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x} \qquad x \to +\infty \tag{3.2}$$

It is customary to let the scattering data, $a(\zeta, t)$, $b(\zeta, t)$, $\bar{a}(\zeta, t)$ and $\bar{b}(\zeta, t)$ be the coefficients relating these two sets of linearly independent solutions

$$\phi = a\bar{\psi} + b\psi \to \begin{pmatrix} a\, e^{-i\zeta x} \\ b\, e^{i\zeta x} \end{pmatrix} \quad \text{as } x \to +\infty$$

$$\bar{\phi} = \bar{b}\bar{\psi} - \bar{a}\psi \to \begin{pmatrix} \bar{b}\, e^{-i\zeta x} \\ -\bar{a}\, e^{i\zeta x} \end{pmatrix} \quad \text{as } x \to +\infty. \tag{3.3}$$

The coefficients $a(\zeta, t)$, $b(\zeta, t)$, $\bar{a}(\zeta, t)$, $\bar{b}(\zeta, t)$ are given by the Wronskians of the solutions ϕ, $\bar{\phi}$, ψ, $\bar{\psi}$:

$$\begin{aligned} a &= W(\phi, \psi) \\ b &= -W(\phi, \bar{\psi}) \\ \bar{a} &= W(\bar{\phi}, \bar{\psi}) \\ \bar{b} &= W(\bar{\phi}, \psi), \end{aligned} \tag{3.4}$$

where $W(u, v) = u_1 v_2 - u_2 v_1$. Since $W(\phi, \bar{\phi}) = -1$, $a\bar{a} + b\bar{b} = 1$. We show in Section IV that $a(\zeta, t)$ can be analytically extended into the upper half plane, $\operatorname{Im} \zeta > 0$, and $\bar{a}(\zeta, t)$ can be extended into the lower half plane, $\operatorname{Im} \zeta < 0$. Indeed, the discrete eigenvalues $\{\zeta_k\}_{k=1}^N$ of (2.1) in the upper half plane ($\operatorname{Im} \zeta > 0$) are given by the zeros of $a(\zeta, t)$, at which $\phi(\zeta_k, t) = b_k(t)\psi(\zeta_k, t)$. Similarly, the zeros of $\bar{a}(\zeta, t)$ in the lower half plane ($\operatorname{Im} \zeta < 0$) are also eigenvalues. At these zeros

$\bar{\phi}_k(\bar{\zeta}_k, t) = \bar{b}_k(t)\bar{\psi}_k(\bar{\zeta}_k, t)$. In general $b(\zeta, t)$ and $\bar{b}(\zeta, t)$ are only defined on the real ζ axis. However, if $q(x, t)$ and $r(x, t)$ have sufficient exponential decay as $x \to \pm\infty$, the regions of analyticity can be extended (see Section IV), and $b_k(t)$ (and $\bar{b}_k(t)$) are the values of $b(\zeta, t)$ (and $\bar{b}(\zeta, t)$) at the eigenvalue $\zeta_k(\bar{\zeta}_k)$.

Because of our choice of normalization in (3.1), we have assumed, (without loss of generality) that B, C in (2.3) tend to zero as $x \to -\infty$. From (2.3a), we see that $A(x, t, \zeta)$ tends to a constant at $x = -\infty$ and it is convenient to set again (with no loss in generality)

$$\lim_{x \to -\infty} A(x, t, \zeta) = A_-(\zeta) \tag{3.5}$$

where $A_-(\zeta)$ is an arbitrary function of ζ. Since it is $\phi\, e^{A_- t}$ and $\bar{\phi}\, e^{-A_- t}$ which satisfy the time equations (2.2), we have

$$\phi_t = \begin{pmatrix} A - A_- & B \\ C & -A - A_- \end{pmatrix}\phi \tag{3.6a}$$

and

$$\bar{\phi}_t = \begin{pmatrix} A + A_- & B \\ C & -A + A_- \end{pmatrix}\bar{\phi}. \tag{3.6b}$$

Using (3.6), the time evolution of the scattering data is given by

$$\begin{aligned} a_t &= (A_+ - A_-)a + B_+ b \\ b_t &= C_+ a - (A_+ + A_-)b \\ \bar{a}_t &= -(A_+ - A_-)\bar{a} - C_+\bar{b} \\ \bar{b}_t &= -B_+\bar{a} + (A_+ + A_-)\bar{b} \end{aligned} \tag{3.7}$$

where

$$A_+ = \lim_{x \to +\infty} A, \quad B_+ = \lim_{x \to +\infty} B\, e^{2i\zeta x} \quad \text{and} \quad C_+ = \lim_{x \to +\infty} C\, e^{-2i\zeta x}.$$

Consider the particular solutions discussed in Section II. In these cases, (see (2.6), and (2.13)), $A_+ = A_-$, $B_+ = C_+ = 0$ and thus equations (3.7) become completely separable and indeed trivial to solve:

$$\begin{aligned} a(\zeta, t) &= a(\zeta, 0) \\ b(\zeta, t) &= b(\zeta, 0)\, e^{-2A_-(\zeta)t} \\ \bar{a}(\zeta, t) &= \bar{a}(\zeta, 0) \\ \bar{b}(\zeta, t) &= \bar{b}(\zeta, 0)\, e^{2A_-(\zeta)t}. \end{aligned} \tag{3.8}$$

In general the situation is more complicated. To solve (3.7), it is necessary to find the general solution of (2.3) and determine A_+, B_+ and C_+. In Appendix 1.1 we sketch a method by which one can solve (2.3). We note that the homogeneous solutions are simply quadratic multiples of the fundamental solutions ϕ, $\bar{\phi}$.

The results can be concisely written in terms of the bilinear form,

$$I(u, v) = \int_{-\infty}^{\infty} (-q_t u_2 v_2 + r_t u_1 v_1)\, dx. \tag{3.9}$$

We may write, using (3.3),

$$\begin{aligned} A_+ &= -I(\psi, \bar{\psi}) + A_-(a\bar{a} - b\bar{b}) \\ B_+ &= -I(\psi, \psi) + 2a\bar{b}A_- \\ C_+ &= I(\bar{\psi}, \bar{\psi}) + 2\bar{a}bA_-. \end{aligned} \tag{3.10}$$

From (3.10), and the inverse relations to (3.3),

$$\begin{aligned} \psi &= -a\phi + \bar{b}\bar{\phi} \\ \bar{\psi} &= b\bar{\phi} + \bar{a}\phi \end{aligned} \tag{3.11}$$

(3.7) can be put in the form,

$$\begin{aligned} a_t &= -I(\phi, \psi) \\ b_t &= I(\phi, \bar{\psi}) \\ \bar{a}_t &= -I(\bar{\phi}, \bar{\psi}) \\ \bar{b}_t &= -I(\bar{\phi}, \psi). \end{aligned} \tag{3.12}$$

In particular we are interested in the time development of the scattering data used in (4.40) and (4.41). Using (3.7) and (3.10) the scattering data evolve according to:

$$\begin{aligned} \left(\frac{b}{a}\right)_t &= \left(\frac{b}{a}\right)\frac{I(\phi, \phi)}{ab} \\ \left(\frac{\bar{b}}{\bar{a}}\right)_t &= \frac{\bar{b}}{\bar{a}}\,\frac{I(\bar{\phi}, \bar{\phi})}{\bar{a}\bar{b}}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \left(\frac{\bar{b}}{a}\right)_t &= \left(\frac{\bar{b}}{a}\right)\frac{I(\psi, \psi)}{a\bar{b}} \\ \left(\frac{b}{\bar{a}}\right)_t &= \left(\frac{b}{\bar{a}}\right)\frac{I(\bar{\psi}, \bar{\psi})}{\bar{a}b}. \end{aligned} \tag{3.14}$$

To this point, we have made no assumptions concerning the evolution of q and r except for the fairly weak condition that the integrals $I(u, v)$ are defined. In principle, for any q_t and r_t, we should be able to compute the time development of the scattering data from one time step to the next and from the inverse scattering equation of Section IV determine $q(x, t)$ and $r(x, t)$ at later times.

However, at present we are more interested in being able to find some analytic expressions relevant to the determination of the evolution equation. It is clear that if we choose, for arbitrary $\Omega(\zeta)$ and $\bar{\Omega}(\zeta)$,

$$I(\psi, \psi) = 2\Omega(\zeta)a\bar{b}, \tag{3.15}$$

and

$$I(\bar{\psi}, \bar{\psi}) = -2\bar{\Omega}(\zeta)\bar{a}b, \tag{3.16}$$

then we linearize the equations (3.14). We may write (3.15) in the form

$$\int_{-\infty}^{\infty} [(r_t + 2\Omega(\zeta)r)\psi_1^2 + (-q_t + 2\Omega(\zeta)q)\psi_2^2]\, dx = 0 \tag{3.17}$$

by noting

$$a\bar{b} = -\psi_1\psi_2|_{-\infty}^{\infty} = -\int_{-\infty}^{\infty} (q\psi_2^2 + r\psi_1^2)\, dx.$$

As may be verified from (2.1) the vector $\Psi = (\psi_1^2, \psi_2^2)^T$, where the superscript T indicates the transpose, satisfies the equation

$$L\Psi = \zeta\Psi \tag{3.18}$$

where

$$L = \frac{1}{2i}\begin{pmatrix} -\dfrac{\partial}{\partial x} - 2q\displaystyle\int_x^{\infty} dy \cdot r(y) & -2q\displaystyle\int_x^{\infty} dy\, q(y) \\ 2r\displaystyle\int_x^{\infty} dy\, r(y) & \dfrac{\partial}{\partial x} + 2r\displaystyle\int_x^{\infty} dy\, q(y) \end{pmatrix}. \tag{3.19}$$

If we define

$$u = (r, q)^T \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we may rewrite (3.17) as

$$\int_{-\infty}^{\infty} (\sigma_3 u_t + 2u\Omega(\zeta)) \cdot \Psi\, dx = 0. \tag{3.20}$$

But if $\Omega(\zeta)$ is an entire function

$$\Omega(\zeta)\Psi = \Omega(L)\Psi. \tag{3.21}$$

Defining the adjoint operator L^+,

$$L^+ = \frac{1}{2i}\begin{pmatrix} \dfrac{\partial}{\partial x} - 2r\displaystyle\int_{-\infty}^{x} dy\, q & \cdot & 2r\displaystyle\int_{-\infty}^{x} dy\, r \\ -2q\displaystyle\int_{-\infty}^{x} dy\, q & \cdot & -\dfrac{\partial}{\partial x} + 2q\displaystyle\int_{-\infty}^{x} dy\, r \end{pmatrix} \tag{3.22}$$

we find (3.20) can be written

$$\int_{-\infty}^{\infty} (\sigma_3 u_t + 2\Omega(L^+)u) \cdot \Psi\, dx = 0. \tag{3.23}$$

We note that (3.16) gives a similar expression

$$\int_{-\infty}^{\infty} (\sigma_3 u_t + 2\bar{\Omega}(L^+)u) \cdot \bar{\Psi}\, dx = 0, \bar{\Psi} = (\bar{\psi}_1^2, \bar{\psi}_2^2)^T. \tag{3.24}$$

In general, then, (3.15–3.16) (and (3.23, 3.24)) give the projection of the vector $\sigma_3 u_t$ into the adjoint eigenfunctions of the equation $L\Psi = \zeta\Psi$. In the special case

$\Omega = \bar{\Omega}$, then in order to satisfy (3.23, 3.24), it is sufficient that

$$\sigma_3 u_t + 2\Omega(L^+)u = 0. \tag{3.25}$$

Without loss of generality we may take $\Omega(\zeta) = A_-(\zeta)$, and find

$$\sigma_3 u_t + 2A_-(L^+)u = 0. \tag{3.26}$$

Equation (3.26) is the nonlinear evolution equation whose linearized dispersion relation is defined by $A_-(\zeta)$ and which can be solved by use of (2.1). For example, if

$$\Omega(\zeta) = A_-(\zeta) = -2i\zeta^2, \tag{3.27}$$

(3.26) yields

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} + i\begin{pmatrix} r_{xx} - 2qr^2 \\ q_{xx} - 2q^2 r \end{pmatrix} = 0 \tag{3.28}$$

which is (2.10). Using (3.26), it also follows that

$$I(\psi, \bar{\psi}) = -2A_-(\zeta)b\bar{b}, \tag{3.29}$$

and from (3.3), that

$$I(\phi, \phi) = -2A_-(\zeta)ab \tag{3.30a}$$

$$I(\bar{\phi}, \bar{\phi}) = 2A_-(\zeta)\bar{a}\bar{b}. \tag{3.30b}$$

From (3.10), note that

$$A_+ = A_-, \qquad B_+ = C_+ = 0 \tag{3.31}$$

and therefore the inverse problem can use either set of scattering data S_+ or S_- (see Section IV). In Section VII, we show that (3.31) leads to the existence of an infinite sequence of integrated densities $\{C_n\}_{n=1}^{\infty}$ which are motion invariants. The first three are:

$$C_1 = \int qr\,dx, \qquad \int C_2 = \tfrac{1}{2}(rq_x - r_x q)\,dx, \qquad C_3 = \int (q_x r_x + q^2 r^2)\,dx. \tag{3.32}$$

We also note here (and discuss in depth in Appendix (3) that when $q \to 0$ as $|x| \to \infty$ and $r = -1$, similar methods yield the evolution equations associated with the Schroedinger equation as the eigenvalue problem. The results can be written concisely:

$$q_t + \hat{C}(4L_s^+)q_x = 0, \tag{3.33}$$

where the operator

$$L_s^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}q_x \int_x^{\infty} dy, \tag{3.34}$$

and $\hat{C}(k^2) = \omega/k$, ω being the dispersion relation of the linearized equation. One can very readily verify that $\omega = -k^3$ yields

$$q_t + q_{xxx} + 6qq_x = 0, \tag{3.35}$$

which is (2.8).

We may extend these ideas by allowing the dispersion relation $\Omega(\zeta)$ to be a ratio of entire functions, say $\Omega_1(\zeta)/\Omega_2(\zeta)$. Then the analogue to (3.25) is simply

$$\Omega_2(L^+)\sigma_3\mathbf{u}_t + 2\Omega_1(L^+)\mathbf{u} = 0. \tag{3.36}$$

In order to appreciate the complexity of these equations and the need for writing them in an alternative form, a few special examples are listed.

If $\Omega = i\alpha/2(\zeta - \hat{\zeta}_1)$, then (3.36) becomes

$$\frac{1}{2i}\begin{pmatrix} r_{xt} - 2r\int_{-\infty}^{x} (qr)_t - 2i\hat{\zeta}_1 r_t \\ q_{xt} - 2q\int_{-\infty}^{x} (qr)_t + 2i\hat{\zeta}_1 q_t \end{pmatrix} = -i\alpha\begin{pmatrix} r \\ q \end{pmatrix}. \tag{3.37}$$

An equation of particular interest arising from the class of equations (3.33) when $\hat{C}(k^2) = 1/(1 + k^2)$ is

$$q_t - q_{xxt} - 4qq_t + 2q_x\int_x^{\infty} q_t\,dy + q_x = 0, \tag{3.38}$$

which reduces to the Korteweg deVries equation in the long wave, small amplitude limit and has the desirable properties of the PBBM (Peregrine, 1966, Benjamin, Bona and Mahoney (1972) equation in that it responds feebly to short waves. It has of course the additional property that it is exactly solvable!

Considerable care must be exercised in interpretating (3.36–3.38) and the question as to the sense in which (3.36) describes the evolution of u must be examined. Considering (3.36) as a linear equation for $\sigma_3 u_t$ where u is known we ask: when does (3.36) have an unique solution? For the class of problems discussed in this paper, the required condition is that the reflection coefficients b and $\bar{b}$ are zero at all *real* poles of the dispersion relation $\Omega(\zeta)$. More importantly, perhaps, we can construct the inverted form of (3.36), (i.e. write $\sigma_3 u_t = \mathscr{L}\{u\}$) and find that it corresponds to the form in which the evolution equations often arise. This material is discussed in Appendix 4.

We also stress that the specification of the dispersion relation $\Omega(\zeta)$ cannot be entirely arbitrary. If the solution (with time) is to remain continuously dependent on the initial data (i.e. remain stable), certain additional stipulations (which can be clearly seen from the inversion formula (4.38, 39) together with the time equations (3.14)) may be necessary. It will usually turn out that $\Omega(\zeta)$ must be purely imaginary when ζ is real.

A further extension of the analysis is possible and an even wider class of evolution equations can be obtained if we do not choose $\bar{\Omega}(\zeta)$ equal to $\Omega(\zeta)$. The inverse scattering transform still applies even though in these cases there are no motion invariants. This material will be discussed in a later paper. A special case, important physically, is worked out in reference [Ablowitz, Kaup, and Newell, 1974].

IV. The direct and inverse scattering problem for the generalized Zakharov–Shabat eigenvalue problem

In this section, we will make a comprehensive study of the direct and inverse scattering problem for this eigenvalue problem. Throughout this section, we shall emphasize the very close analogy that exists between the IST and the Fourier transform. In part A, we shall first look at the analytic properties of the scattering data and see how these properties are related to the gross structure of the potentials, r and q. Here, we shall see that the "reflection coefficients" are, in effect, simply "nonlinear" Fourier transforms of the potentials. Next, we shall obtain integral representations for the Jost functions, which we shall use in part B for deriving the inverse scattering equations of the Marchenko type, [Marchenko, 1955] and we shall see that these inverse scattering equations are just a generalization of the inverse Fourier transform to nonlinear problems.

Once this has been accomplished, one has at hand all of the necessary equations for solving nonlinear evolution equations by this inverse scattering transform. However, there are further aspects of these equations which we consider to be of sufficient interest for further elaboration and are placed in the appendices. First, from the Marchenko equations, it is immediately obvious that much of the information obtained from the direct scattering problem is redundant, as in the case of the Schroedinger equation. In Appendix 5, we show that all of the information obtainable from the direct scattering problem can be given in terms of smaller sets, which we shall call the "primordial" scattering data (and is referred to by Faddeev [1958] as simply the scattering data). Consequently, for any inversion, only these smaller sets need to be specified, and the totality of the information obtainable from the direct scattering problem is not required. Second, although the Zakharov–Shabat eigenvalue problem is not self-adjoint in general, still it is possible to derive the Marchenko equations from a "gedenken" scattering experiment, as can be done for the Schroedinger equation. All that is required for such a derivation is the completeness relation for the eigenstates. One of the important by-products of our investigations is that we can give the completeness relation (with respect to L_2) for this nonself-adjoint and unbounded operator, as well as indicate how completeness relations can also be obtained for other nonself-adjoint and unbounded eigenvalue problems. This completeness relation is derived in Appendix 6. Such a completeness relation naturally contains imbedded in it, all of the analytical properties which we will be deriving in this section. Consequently, instead of obtaining the inverse scattering equations from the integral representations of the Jost functions, as we do in this section, one can also derive these equations directly from the completeness relation. This derivation is much more instructive from a physical point of view since it involves actually doing a "gedenken" scattering experiment, in contrast to taking contours through the complex plane. This derivation will be done in Appendix 7.

A. Analytical properties of the scattering data

The general eigenvalue problem is given by (2.1) on the interval $-\infty < x < \infty$, and we assume r and q to vanish sufficiently rapidly as $x \to \pm\infty$ so that in these limits, the right hand side in (2.1) can be neglected. Let ϕ, $\bar{\phi}$, ψ and $\bar{\psi}$ be the Jost functions of (2.1) satisfying the boundary conditions in (3.1,2). Clearly ϕ and $\bar{\phi}$ are linearly independent, as are ψ and $\bar{\psi}$. Therefore, for ζ real, we must have

$$\phi(\zeta, x) = a(\zeta)\bar{\psi}(\zeta, x) + b(\zeta)\psi(\zeta, x), \tag{4.1a}$$

$$\bar{\phi}(\zeta, x) = -\bar{a}(\zeta)\psi(\zeta, x) + \bar{b}(\zeta)\bar{\psi}(\zeta, x), \tag{4.1b}$$

which defines a, $\bar{a}$, b and $\bar{b}$. From (2.1), if $u(\zeta, x)$ and $v(\zeta, x)$ are solutions of (2.1), then

$$\frac{dW(u, v)}{dx} = 0, \tag{4.2}$$

where

$$W(u, v) \equiv u_1(\zeta, x)v_2(\zeta, x) - u_2(\zeta, x)v_1(\zeta, x). \tag{4.3}$$

Thus we also have the relations (3.4) and the condition $W(\bar{\phi}, \phi) = 1$ implies

$$\bar{a}(\zeta)a(\zeta) + \bar{b}(\zeta)b(\zeta) = 1. \tag{4.4}$$

Then the inverse of (4.1) is simply

$$\psi(\zeta, x) = -a(\zeta)\bar{\phi}(\zeta, x) + \bar{b}(\zeta)\phi(\zeta, x), \tag{4.5a}$$

$$\bar{\psi}(\zeta, x) = \bar{a}(\zeta)\phi(\zeta, x) + b(\zeta)\bar{\phi}(\zeta, x). \tag{4.5b}$$

From (2.1) and (3.1), we have the following integral equations for ϕ:

$$e^{i\zeta x}\phi_1(x) = 1 + \int_{-\infty}^{x} M(\zeta, x, y)\, e^{i\zeta y}\phi_1(y)\, dy \tag{4.6a}$$

$$e^{i\zeta x}\phi_2(x) = \int_{-\infty}^{x} e^{2i\zeta(x-y)}r(y)\, e^{i\zeta y}\phi_1(y)\, dy \tag{4.6b}$$

where

$$M(\zeta, x, y) \equiv r(y)\int_{y}^{x} e^{2i\zeta(z-y)}q(z)\, dz \tag{4.7}$$

Under suitable conditions, we can extend ϕ into the upperhalf of the ζ-plane ($\zeta = \xi + i\eta$) by the above equations. To see this, let

$$R_n(x) \equiv \int_{-\infty}^{x} |y^n| \cdot |r(y)|\, dy \tag{4.8a}$$

$$Q_n(x) \equiv \int_{-\infty}^{x} |y^n| \cdot |q(y)|\, dy \tag{4.8b}$$

where we now assume r and q to vanish sufficiently rapidly as $x \to -\infty$ for at least some of these integrals to exist when $n > 0$. For $\eta \geq 0$, we have

$$\begin{aligned} |e^{i\zeta x}\phi_1(x)| &\leq 1 + \int_{-\infty}^{x} Q_0'(z)\, dz \int_{-\infty}^{z} R_0'(y) |e^{i\zeta y}\phi_1(y)|\, dy \\ &\leq 1 + R_0(x)Q_0(x) + \frac{[R_0(x)Q_0(x)]^2}{(2!)^2} + \frac{[R_0(x)Q_0(x)]^3}{(3!)^2} + \cdots, \end{aligned}$$

or

$$|e^{i\zeta x}\phi_1(x)| \leq I_0(S(x)), \tag{4.9}$$

where $S(x) = 2(R_0(x)Q_0(x))^{1/2}$ and $I_0(S)$ is the Bessel function of order zero and imaginary argument. Thus, by (3.2), (3.3) and (4.1a), upon letting $x \to +\infty$, we have that $a(\zeta)$ is bounded in the upper half ζ-plane ($\eta \geq 0$) if $R_0(\infty)$ and $Q_0(\infty)$

are finite. Returning to (4.6) we see that the Neumann series solution of

$$e^{i\zeta x}\phi_1(x) = 1 + \int_{-\infty}^{x} M(\zeta, x, y)\,dy + \int_{-\infty}^{x} M(\zeta, x, y)\,dy \int_{-\infty}^{y} M(\zeta, y, z)\,dz + \cdots \tag{4.10}$$

is absolutely convergent in the upper half plane.

Furthermore, one may differentiate (4.10) with respect to ζ and one may then see that this is an analytic function if $\eta > 0$. To be analytic for $\eta = 0$, it is easy to see that simply requiring $R_0(\infty)$ and $Q_0(\infty)$ to be finite is not sufficient. For example, since ζ occurs in an exponential in (4.7), differentiation will bring down a $(z - y)$ term, and consequently, for a first differential to exist at $\eta = 0$, r and q must vanish faster than x^{-2} as $x \to \pm\infty$. Upon doing the same for $\bar{\phi}$, ψ, and $\bar{\psi}$ as for ϕ, we have the following theorem:

If

$$\left.\begin{aligned} R_0(\infty) &< \infty, \\ Q_0(\infty) &< \infty, \end{aligned}\right\} \tag{4.11}$$

[defined by (4.8) for $n = 0$]

then $e^{i\zeta x}\phi(\zeta, x)$ and $e^{-i\zeta x}\psi(\zeta, x)$ are analytic functions of ζ when $\eta > 0$, while $e^{-i\zeta x}\bar{\phi}(\zeta, x)$ and $e^{i\zeta x}\bar{\psi}(\zeta, x)$ are analytic functions of ζ when $\eta < 0$. Also, when $\eta = 0$, the above four functions are bounded. Furthermore, if for a given integer n,

$$\left.\begin{aligned} R_l(\infty) &< \infty, \\ Q_l(\infty) &< \infty, \end{aligned}\right\} \quad (l = 0, 1, 2, \ldots, n), \tag{4.12}$$

then these four functions are also n-fold differentiable (with respect to ζ) at $\eta = 0$. If (4.12) is true for all n, then the range of analyticity will include the real ζ-axis ($\eta = 0$).

As a corollary, we immediately have from (3.4) that:

$a(\zeta)$ is an analytic function of ζ if $\eta > 0$ and $\bar{a}(\zeta)$ is an analytic function of ζ if $\eta < 0$ when (4.11) is satisfied. If also, (4.12) is true for all n, then both $a(\zeta)$ and $\bar{a}(\zeta)$ are analytic when $\eta = 0$ as well.

When more stringent conditions can be placed on r and q, one can do better. By identical techniques, one can show the following theorem:

If there exist an $\hat{R}$, $\hat{Q}$ and K, all finite and greater than zero, such that for all x

$$\left.\begin{aligned} |r(x)| &\le \hat{R}\, e^{-2K|x|}, \\ |q(x)| &\le \hat{Q}\, e^{-2K|x|}, \end{aligned}\right\} \tag{4.13}$$

then $e^{i\zeta x}\phi(\zeta, x)$ and $e^{-i\zeta x}\psi(\zeta, x)$ are analytic functions of ζ when $\eta > -K$, while $e^{-i\zeta x}\bar{\phi}(\zeta, x)$ and $e^{i\zeta x}\bar{\psi}(\zeta, x)$ are analytic functions of ζ when $\eta < +K$.

As an immediate corollary, we have from (3.4):

$a(\zeta)$ is an analytic function of ζ when $\eta > -K$, $\bar{a}(\zeta)$ when $\eta < +K$, and both $b(\zeta)$ and $\bar{b}(\zeta)$ when $+K > \eta > -K$.

We note when both r and q are on compact support, the K in (4.13) can be chosen as large as desired. Thus a second corollary is:

When r and q are on compact support, and (4.11) is true then $e^{i\zeta x}\phi(\zeta, x)$, $e^{-i\zeta x}\bar{\phi}(\zeta, x)$, $e^{-i\zeta x}\psi(\zeta, x)$ and $e^{i\zeta x}\bar{\psi}(\zeta, x)$ are entire functions of ζ. Thus $a(\zeta)$, $\bar{a}(\zeta)$, $b(\zeta)$, and $\bar{b}(\zeta)$ are also entire functions of ζ.

Returning to (4.6,7), we note for ζ in the upper half plane, as $|\zeta| \to \infty$, we have the asymptotic series

$$\phi_1 e^{i\zeta x} \to 1 - \frac{1}{2i\zeta}\int_{-\infty}^{x} r(x)q(x)\,dy + O\left(\frac{1}{\zeta^2}\right), \tag{4.14a}$$

$$\phi_2 e^{i\zeta x} \to \frac{-1}{2i\zeta} r(x) + O\left(\frac{1}{\zeta^2}\right), \tag{4.14b}$$

and similarly,

$$\psi_1 e^{-i\zeta x} \to \frac{1}{2i\zeta} q(x) + O\left(\frac{1}{\zeta^2}\right), \tag{4.15a}$$

$$\psi_2 e^{-i\zeta x} \to 1 - \frac{1}{2i\zeta}\int_{x}^{\infty} r(y)q(y)\,dy + O\left(\frac{1}{\zeta^2}\right), \tag{4.15b}$$

while for ζ in the lower half plane, still as $|\zeta| \to \infty$,

$$\bar{\phi}_1 e^{-i\zeta x} \to \frac{-1}{2i\zeta} q(x) + O\left(\frac{1}{\zeta^2}\right), \tag{4.16a}$$

$$\bar{\phi}_2 e^{-i\zeta x} \to -1 - \frac{1}{2i\zeta}\int_{-\infty}^{x} q(y)r(y)\,dy + O\left(\frac{1}{\zeta^2}\right), \tag{4.16b}$$

$$\bar{\psi}_1 e^{i\zeta x} \to 1 + \frac{1}{2i\zeta}\int_{x}^{\infty} q(y)r(y)\,dy + O\left(\frac{1}{\zeta^2}\right), \tag{4.17a}$$

$$\bar{\psi}_2 e^{i\zeta x} \to \frac{-1}{2i\zeta} r(x) + O\left(\frac{1}{\zeta^2}\right). \tag{4.17b}$$

Thus in each respective half-plane, we have, as $|\zeta| \to \infty$,

$$a(\zeta) \to 1 - \frac{1}{2i\zeta}\int_{-\infty}^{\infty} q(y)r(y)\,dy + O\left(\frac{1}{\zeta^2}\right), \tag{4.18a}$$

$$\bar{a}(\zeta) \to 1 + \frac{1}{2i\zeta}\int_{-\infty}^{\infty} q(y)r(y)\,dy + O\left(\frac{1}{\zeta^2}\right). \tag{4.18b}$$

At this point, it is worthwhile to pause in our rigor and to consider what is happening. We first specify the potentials r and q and then we solve for the "scattering" coefficients, a, $\bar{a}$, b, and $\bar{b}$. Thus, this is simply a "mapping" process whereby we are mapping r and q into these scattering coefficients. To see what this mapping process corresponds to, let us look at the case when r and q become infinitesimally small. From (4.9), we see that we can make $e^{i\zeta x}\phi_1(\zeta, x)$ as close to unity for all x as we desire, and consequently a and $\bar{a}$ also approach unity as indicated by (4.18). Using this result to solve (4.6b) shows that in this limit [from

(4.1) and (3.2)],

$$b(\zeta) \to \tilde{r}(2\zeta), \tag{4.19a}$$

where $\tilde{r}(k)$ is just the Fourier transform of $r(x)$. Similarly

$$\bar{b}(\zeta) \to -\tilde{q}(-2\zeta). \tag{4.19b}$$

Thus, this mapping is just simply the mapping of r and q into their Fourier transforms when the potentials are infinitesimally small. Of course, the condition for being able to map a function into its Fourier transform everywhere is simply (4.11). We shall return to this case in part B after we derive the inverse scattering equations and will show there that in this limit, these equations yield the inverse Fourier transform.

Whenever r and q are not infinitesimal, (2.1) can also possess bound states. These occur whenever a has a zero in the upper half plane ($\eta > 0$) or whenever $\bar{a}$ has a zero in the lower half plane ($\eta < 0$). If we designate the zeros of a by ζ_k ($k = 1, 2, \ldots, N$) where N is the number of these bound states, then at $\zeta = \zeta_k$, from (3.4), ϕ is proportional to ψ, or

$$\phi = b_k \psi, \tag{4.20a}$$

where in the case of r and q on compact support, $b_k \equiv b(\zeta_k)$. Also, whenever $\bar{a}$ is zero in the lower half plane, we have further bound states. We designate these by $\bar{\zeta}_k$ ($k = 1, 2, \ldots, \bar{N}$) with $\bar{N}$ being the total number of these bound states. When $\zeta = \bar{\zeta}_k$,

$$\bar{\phi} = \bar{b}_k \bar{\psi}, \tag{4.20b}$$

and if r and q are on compact support, $\bar{b}_k \equiv \bar{b}(\bar{\zeta}_k)$. Normally, N and $\bar{N}$ are finite; however, exceptions can occur. (See Section VI.) In this section we shall assume that N and $\bar{N}$ are always finite.

Unlike the Schroedinger equation, this eigenvalue problem can have zeros of any order for a and $\bar{a}$. (For the Schroedinger equation, all zeros must be simple as a consequence of being self-adjoint.) However, these cases may be analyzed as the limit of the case where all zeros are simple. For example, a double zero in a at ζ_1 is simply the limit of letting a have two simple zeros at ζ_1 and ζ_2 and allowing $\zeta_2 \to \zeta_1$. More will be said about these special cases in Appendix 6.

Whenever r is linearly related to q or q^*, simplifications occur. First consider the case where

$$r = \alpha q, \tag{4.21}$$

where α is any nonzero, finite, complex scalar constant. In this case, one can show

$$\bar{\psi}(\zeta, x) = S\psi(-\zeta, x), \tag{4.22a}$$

$$\bar{\phi}(\zeta, x) = \frac{-1}{\alpha} S\phi(-\zeta, x), \tag{4.22b}$$

where

$$S = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}. \tag{4.23}$$

Consequently,

$$\bar{a}(\zeta) = a(-\zeta), \tag{4.24a}$$

$$\bar{b}(\zeta) = \frac{-1}{\alpha} b(-\zeta), \tag{4.24b}$$

and the zeros of a and $\bar{a}$ are paired such that

$$\bar{N} = N, \tag{4.25a}$$

$$\bar{\zeta}_k = -\zeta_k \ (k = 1, 2, \ldots, N), \tag{4.25b}$$

$$\bar{b}_k = -\frac{1}{\alpha} b_k. \tag{4.25c}$$

In the case where

$$r = \alpha q^*, \tag{4.26}$$

with α as a nonzero, finite, and *real* constant, we have

$$\bar{\psi}(\zeta, x) = S\psi^*(\zeta^*, x), \tag{4.27a}$$

$$\bar{\phi}(\zeta, x) = \frac{-1}{\alpha} S\phi^*(\zeta^*, x), \tag{4.27b}$$

which gives

$$\bar{a}(\zeta) = a^*(\zeta^*), \tag{4.28a}$$

$$\bar{b}(\zeta) = -\frac{1}{\alpha} b^*(\zeta^*). \tag{4.28b}$$

Again, the zeros of a and $\bar{a}$ are paired, but in a different manner, given by

$$\bar{N} = N, \tag{4.29a}$$

$$\bar{\zeta}_k = \zeta_k^* \ (k = 1, 2, \ldots, N), \tag{4.29b}$$

$$\bar{b}_k = -\frac{1}{\alpha} b_k^*. \tag{4.29c}$$

Hence, if (4.21) and (4.26) both hold (r and r^* both proportional to q), then ζ_k is either pure imaginary or $-\zeta_k^*$ is also another eigenvalue. This will be discussed more fully in Section V.

B. Method of inversion

First, we will obtain integral representations for the four Jost functions defined by (2.1), from which we will obtain the inversion equations of the Marchenko type. For simplicity, we will assume r and q to be on compact support so that the solutions of (2.1) and the scattering data will be entire functions of ζ. One can easily show that all of the following results hold in the more general case of non-compact support, if all contour integrals are reduced to integrals along the real axis plus all contributions due to any poles. We define the contour C to be the contour in the complex ζ-plane, starting from $\zeta = -\infty + i0^+$, passing *over all* zeros of $a(\zeta)$, and ending at $\zeta = +\infty + i0^+$. Similarly, $\bar{C}$ is the contour starting from $\zeta = -\infty + i0^-$, passing *under all* zeros of $\bar{a}(\zeta)$, and ending at $\zeta = +\infty + i0^-$.

Consider the integral [Zakharov and Shabat, 1972]

$$\oint \frac{d\zeta'}{a(\zeta')} \frac{\phi(\zeta', x)\, e^{i\zeta' x}}{\zeta' - \zeta},$$

for ζ below C. From (4.14, 18), we find that its value is $-i\pi\binom{1}{0}$. Using (4.1a), and closing the contour for the integral containing $\bar{\psi}$ in the lower ζ-plane, upon using (4.17), gives

$$\bar{\psi}(\zeta, x)\, e^{i\zeta x} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi i}\oint \frac{d\zeta'}{\zeta' - \zeta} \frac{b(\zeta')}{a} \psi(\zeta', x)\, e^{i\zeta' x}, \tag{4.30}$$

for ζ below C. Similarly, considering the integral

$$\oint \frac{d\zeta'}{\zeta' - \zeta} \frac{\bar{\phi}(\zeta', x)\, e^{-i\zeta' x}}{a(\zeta')},$$

for ζ above $\bar{C}$, gives

$$\psi(\zeta, x)\, e^{-i\zeta x} = \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\pi i}\oint \frac{d\zeta'}{\zeta' - \zeta} \frac{\bar{b}(\zeta')}{\bar{a}(\zeta')} \bar{\psi}(\zeta', x)\, e^{-i\zeta' x}. \tag{4.31}$$

Likewise, replacing $\phi\, e^{i\zeta x}$ and $\bar{\phi}\, e^{-i\zeta x}$ by $\psi\, e^{-i\zeta x}$ and $\bar{\psi}\, e^{i\zeta x}$ respectively in the above contour integrals, gives also

$$\bar{\phi}(\zeta, x)\, e^{-i\zeta x} = -\begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{2\pi i}\oint \frac{d\zeta'}{\zeta' - \zeta} \frac{\bar{b}(\zeta')}{a(\zeta')} \phi(\zeta', x)\, e^{-i\zeta' x}, \tag{4.32}$$

$$\phi(\zeta, x)\, e^{i\zeta x} = \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2\pi i}\oint \frac{d\zeta'}{\zeta' - \zeta} \frac{b(\zeta')}{\bar{a}(\zeta')} \bar{\phi}(\zeta', x)\, e^{i\zeta' x}, \tag{4.33}$$

when ζ lies between the contours C and $\bar{C}$.

Now, let us first assume (proof to follow) that ψ, $\bar{\psi}$, ϕ, and $\bar{\phi}$ can be represented as

$$\psi(\zeta, x) = \begin{pmatrix}0\\1\end{pmatrix} e^{i\zeta x} + \int_x^\infty K(x, s)\, e^{i\zeta s}\, ds, \tag{4.34}$$

$$\bar{\psi}(\zeta, x) = \begin{pmatrix}1\\0\end{pmatrix} e^{-i\zeta x} + \int_x^\infty \bar{K}(x, s)\, e^{-i\zeta s}\, ds, \tag{4.35}$$

$$\phi(\zeta, x) = \begin{pmatrix}1\\0\end{pmatrix} e^{-i\zeta x} - \int_{-\infty}^x L(x, s)\, e^{-i\zeta s}\, ds, \tag{4.36}$$

$$\bar{\phi}(\zeta, x) = -\begin{pmatrix}0\\1\end{pmatrix} e^{i\zeta x} - \int_{-\infty}^x \bar{L}(x, s)\, e^{i\zeta s}\, ds, \tag{4.37}$$

where K, $\bar{K}$, L and $\bar{L}$ are column vectors. Inserting these into (4.30–33) gives equations of the Marchenko type, upon taking a Fourier transform. These are

$$\bar{K}(x, y) + \begin{pmatrix}0\\1\end{pmatrix}F(x + y) + \int_x^\infty K(x, s)F(s + y)\,ds = 0, (y > x) \qquad (4.38a)$$

$$K(x, y) - \begin{pmatrix}1\\0\end{pmatrix}\bar{F}(x + y) - \int_x^\infty \bar{K}(x, s)\bar{F}(s + y)\,ds = 0, (y > x) \qquad (4.38b)$$

$$\bar{L}(x, y) + \begin{pmatrix}1\\0\end{pmatrix}G(x + y) - \int_{-\infty}^x L(x, s)G(s + y)\,ds = 0, (x > y) \qquad (4.39a)$$

$$L(x, y) + \begin{pmatrix}0\\1\end{pmatrix}\bar{G}(x + y) + \int_{-\infty}^x \bar{L}(x, s)\bar{G}(s + y)\,ds = 0, (x \geq y) \qquad (4.39b)$$

where

$$F(z) \equiv \frac{1}{2\pi}\oint \frac{b(\zeta)}{a(\zeta)} e^{i\zeta z}\,d\zeta, \qquad (4.40a)$$

$$\bar{F}(z) \equiv \frac{1}{2\pi}\oint \frac{\bar{b}(\zeta)}{\bar{a}(\zeta)} e^{-i\zeta z}\,d\zeta, \qquad (4.40b)$$

$$G(z) \equiv \frac{1}{2\pi}\oint \frac{\bar{b}(\zeta)}{a(\zeta)} e^{-i\zeta z}\,d\zeta, \qquad (4.41a)$$

$$\bar{G}(z) \equiv \frac{1}{2\pi}\oint \frac{b(\zeta)}{\bar{a}(\zeta)} e^{i\zeta z}\,d\zeta. \qquad (4.41b)$$

From (4.14–17) and (4.34–37), we have

$$K_1(x, x) = -\bar{L}_1(x, x) = -\tfrac{1}{2}q(x), \qquad (4.42a)$$

$$K_2(x, x) = \bar{K}_1(x, x) = \frac{1}{2}\int_x^\infty q(y)r(y)\,dy, \qquad (4.42b)$$

$$L_1(x, x) = -\bar{L}_2(x, x) = \frac{1}{2}\int_{-\infty}^x q(y)r(y)\,dy, \qquad (4.42c)$$

$$L_2(x, x) = \bar{K}_2(x, x) = \tfrac{1}{2}r(x). \qquad (4.42d)$$

As already mentioned at the first of this section. the results for noncompact support can be obtained by reducing the above contour integrals to integrals along the real axis plus contributions from any poles. For example, (4.40a) and (4.41a) become (when a has only simple zeros)

$$F(z) = -i\sum_{k=1}^N \frac{b_k}{a_k'} e^{i\zeta_k z} + \frac{1}{2\pi}\int_{-\infty}^\infty \frac{b(\xi)}{a(\xi)} e^{i\xi z}\,d\xi, \qquad (4.43)$$

$$G(z) = -i\sum_{k=1}^N \frac{e^{-i\zeta_k z}}{b_k a_k'} + \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\bar{b}(\xi)}{a(\xi)} e^{-i\xi z}\,d\xi, \qquad (4.44)$$

where $(a_k')^{-1}$ is the residue of $[a(\zeta)]^{-1}$ at $\zeta = \zeta_k$ $(k = 1, 2, \ldots, N)$. If a has a zero on the real axis $(\eta = 0)$, then the sum in (4.43) has "i" replaced by "$i/2$" for these poles and the integral is replaced by its Cauchy principle value. Note the advantage of the compact notation used in (4.40, 41). To give the general expressions when a has zeros of an arbitrary order would be very long and difficult.

Let us now solve (4.38) when the potentials are infinitesimal. In this limit, we have no bound states and F and $\bar{F}$ in (4.40) degenerates into integrals along the real axis. From (4.19) we have

$$F(z) \cong \tfrac{1}{2}r(z/2),$$
$$\bar{F}(z) \cong -\tfrac{1}{2}q(z/2),$$

giving the solution of (4.38) as

$$\bar{K}(x, y) \cong -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} r\left(\frac{x+y}{2}\right),$$
$$K(x, y) \cong +\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} q\left(\frac{x+y}{2}\right),$$

which does indeed satisfy (4.42) in this limit. Thus the inverse scattering transformation does contain the Fourier transformation as a special case.

Now we will prove that the K, $\bar{K}$, L and $\bar{L}$ assumed in (4.34–37) exist, are unique, and that they are independent of ζ. Starting with (4.34), we first require it to satisfy (2.1). This gives the two equations:

$$\int_x^\infty e^{i\zeta s}[(\partial_x - \partial_s)K_1(x, s) - q(x)K_2(x, s)]\, ds$$
$$- [q(x) + 2K_1(x, x)]\, e^{i\zeta x} + \lim_{s\to\infty} [K_1(x, s)\, e^{i\zeta s}] = 0, \tag{4.45a}$$

$$\int^\infty e^{i\zeta s}[(\partial_x + \partial_s)K_2(x, s) - r(x)K_1(x, s)]\, ds - \lim_{s\to\infty} [K_2(x, s)\, e^{i\zeta s}] = 0. \tag{4.45b}$$

It is necessary and sufficient to have

$$(\partial_x - \partial_s)K_1(x, s) - q(x)K_2(x, s) = 0, \tag{4.46a}$$
$$(\partial_x + \partial_s)K_2(x, s) - r(x)K_1(x, s) = 0, \tag{4.46b}$$

subject to the boundary conditions

$$K_1(x, x) = -\tfrac{1}{2}q(x), \tag{4.47a}$$
$$\lim_{s\to\infty} K(x, s) = 0. \tag{4.47b}$$

To see that a solution of (4.46) exists subject to the boundary conditions (4.47), introduce the coordinates

$$\mu = \tfrac{1}{2}(x + s),$$
$$\nu = \tfrac{1}{2}(x - s), \tag{4.48}$$

and upon transforming to these coordinates, (4.46–47) becomes

$$\partial_\nu K_1(\mu, \nu) - q(\mu + \nu)K_2(\mu, \nu) = 0, \tag{4.49a}$$
$$\partial_\mu K_2(\mu, \nu) - r(\mu + \nu)K_1(\mu, \nu) = 0, \tag{4.49b}$$

$$K_1(\mu, 0) = -\tfrac{1}{2}q(\mu), \tag{4.50a}$$

$$\lim_{\mu - \nu \to \infty} K(\mu, \nu) = 0. \tag{4.50b}$$

From the theory of characteristics, the solution exists and is unique. Similarly, one can show that $\bar{K}$, L and $\bar{L}$ exist and are unique.

Finally, we consider the existence and uniqueness of the solution of the Marchenko equations (4.38 or 39). We will show that either of the restrictions:

$$r(x) = -q^*(x); \tag{4.51}$$

or

$$r(x) = +q^*(x), \tag{4.52a}$$

and

$$Q(\infty) = \int_{-\infty}^{\infty} |q|\, dx < .523 \tag{4.52b}$$

is sufficient to guarantee that the solution of the Marchenko equations is uniquely defined. Neither of these restrictions is necessary, but as seen in Section V, some restrictions of (r, q) are necessary for the existence of solutions of these equations. Requirements which are both necessary and sufficient have not yet been determined.

The homogeneous equations corresponding to (4.38) are $(y > x)$

$$\phi_1(y) + \int_x^{\infty} \phi_2(s)F(s + y)\, ds = 0,$$

$$\phi_2(y) - \int_x^{\infty} \phi_1(s)\bar{F}(s + y)\, ds = 0. \tag{4.53}$$

Suppose $\phi(y) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a solution of (4.53) which vanishes identically for $y < x$. By the Fredholm alternatives, it is sufficient to show that $\phi(y) \equiv 0$. Multiply (4.53) by $[\phi_1^*, \phi_2^*]$, integrate in y and use

$$\int_x^{\infty} |\phi_j(y)|^2\, dy = \int_{-\infty}^{\infty} |\phi_j(y)|^2\, dy.$$

One obtains

$$\int_{-\infty}^{\infty} \left\{ |\phi_1|^2 + |\phi_2|^2 + \int_{-\infty}^{\infty} [\phi_2(s)\phi_1^*(y)F(s + y) - \phi_1(s)\phi_2^*(y)\bar{F}(s + y)]\, ds \right\} dy = 0. \tag{4.54}$$

We consider two special cases. First, if

$$r = -q^*,$$

then it follows from (4.26)–(4.29) that

$$\bar{F}(s + y) = F^*(s + y).$$

Hence, (4.54) becomes

$$\int_{-\infty}^{\infty}\left\{|\phi_1|^2+|\phi_2|^2+2i\,\mathrm{Im}\int_{-\infty}^{\infty}\phi_1^*(y)\phi_2(s)F(s+y)\,ds\right\}dy=0. \qquad (4.55)$$

The real and imaginary parts must both vanish, from which it follows that

$$\phi(y)\equiv 0,$$

and the solution of (4.38) exists and is unique.

Second, if

$$r(x)=+q^*(x), \qquad (4.56)$$

the problem is formally self-adjoint, the spectrum lies on the real axis, and

$$\bar{F}(s+y)=-F^*(s+y).$$

In this case, (4.54) becomes

$$\int_{-\infty}^{\infty}\left\{|\phi_1|^2+|\phi_2|^2+2\,\mathrm{Re}\int_{-\infty}^{\infty}\phi_1^*(y)\phi_2(s)F(s+y)\,ds\right\}dy=0. \qquad (4.57)$$

We require

$$|a(\zeta)|>0, \qquad (\eta\geq 0) \qquad (4.58)$$

from which it follows that there are no discrete eigenvalues on the real axis, hence

$$F(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b}{a}(\xi)\,e^{i\xi z}\,d\xi. \qquad (4.59)$$

The Fourier transform of $\phi_j(y)$ is

$$\hat{\phi}_j(\xi)=\int_{-\infty}^{\infty}\phi_j(y)\,e^{-i\xi y}\,dy, \qquad (4.60)$$

which satisfies Parseval's relation:

$$\int_{-\infty}^{\infty}|\phi_j|^2\,dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{\phi}_j|^2\,d\xi. \qquad (4.61)$$

Substitute (4.59)–(4.61) into (4.57) and reverse the order of integration:

$$\int_{-\infty}^{\infty}\left\{|\hat{\phi}_1(-\xi)|^2+|\hat{\phi}_2^*(\xi)|^2+2\,\mathrm{Re}\left[\frac{b}{a}(\xi)\hat{\phi}_1(-\xi)\hat{\phi}_2^*(\xi)\right]\right\}d\xi=0. \qquad (4.62)$$

If

$$\left|\frac{b}{a}(\xi)\right|<1, \qquad (4.63)$$

then

$$\left|2\,\mathrm{Re}\left[\frac{b}{a}(\xi)\hat{\phi}_1(-\xi)\hat{\phi}_2(\xi)\right]\right|<2|\hat{\phi}_1(-\xi)|\cdot|\hat{\phi}_2^*(\xi)|, \leq|\hat{\phi}_1|^2+|\hat{\phi}_2|^2.$$

Hence the only solution of (4.62) is

$$\phi \equiv 0.$$

Again, it follows that the solution of (4.38) exists and is unique.

To relate (4.63) to the initial data, we use (4.4) and (4.28). With (4.56), (4.63) can be written as

$$|a|^2 > \tfrac{1}{2}, \tag{4.64}$$

which is more stringent than (4.58). This is satisfied if

$$|a(\zeta) - 1| < 1 - \frac{1}{\sqrt{2}}.$$

We use (4.9) (and 4.56), and require

$$|a(\zeta) - 1| \leq I_0(2Q(\infty)) - 1 < 1 - \frac{1}{\sqrt{2}}. \tag{4.65}$$

Thus, it is sufficient that

$$Q(\infty) < .523$$

which is (4.52b).

In this second case (4.52), the original problem (2.1) is a Dirac system, for which existence and uniqueness were shown explicitly in a related problem by Gasymov and Levitan [1966 a, b]. They did not need (4.52b), which suggests it may be unnecessary here as well.

In concluding this section, we shall point out a remaining question and unsolved problem. We have not shown that the solution of the inversion equations (4.38) or (39) is unique. In general, it is *not* since one can explicitly show that homogeneous solutions can exist in the general case (see Section V). However, since (4.49–50) shows that K does exist when (4.11) is true and since the inversion equations follow from (4.30–33), we have that the required solution must be contained in the general solution of the inversion equation.

V. Asymptotic solution of the evolution equation

In previous sections, we have explained the method of inverse scattering transforms, and defined a class of nonlinear evolution equations that can be solved as initial value problems by this method. In this section we solve the integral equations ($y > x$)

$$\begin{aligned} K(x, y; t) - \begin{pmatrix}1\\0\end{pmatrix}\bar{F}(x + y; t) - \int_x^\infty \bar{K}(x, s; t)\bar{F}(s + y; t)\, ds = 0, \\ \bar{K}(x, y; t) + \begin{pmatrix}0\\1\end{pmatrix}F(x + y; t) + \int_x^\infty K(x, s; t)F(s + y; t)\, ds = 0, \end{aligned} \tag{5.1}$$

in order to determine the asymptotic behavior ($t \to \infty$) of the solutions of evolution equations of the form (1.16). The asymptotic solution will be shown to be similar to that of the KdV equation, although there are some important differences. We discuss separately the contribution to the solution from the discrete spectrum, the continuous spectrum and their combination.

A. The discrete spectrum

Consider first the solvability of the system (5.1). An important difference between the Schroedinger equation (1.6) and the scattering problem (1.14) used here is that the system (5.1) need not have a solution, and the solution of the evolution equation can become unbounded after a finite time. We illustrate with an example. Let $q(x, 0)$, $r(x, 0)$ be smooth initial data which satisfy (4.11), and for which the spectrum consists of two discrete eigenvalues, $\zeta(\operatorname{Im}\zeta > 0)$ and $\bar{\zeta}$ $(\operatorname{Im}\bar{\zeta} < 0)$. Then

$$\begin{aligned} F(z, t) &= -ic\, e^{i\zeta z - 2A_0(\zeta)t} \\ \bar{F}(z, t) &= i\bar{c}\, e^{-i\bar{\zeta}z + 2A_0(\bar{\zeta})t}, \end{aligned} \tag{5.2}$$

where $c, \bar{c}$ are constants and $A_0(\zeta)$ is related to the linear dispersion relation. The integral kernel is degenerate in this case, and (5.1) can be solved in closed form. From (4.42), the corresponding solutions of the evolution equations are

$$\begin{aligned} q(x, t) &= -\frac{2i\bar{c}\, e^{2A_0(\bar{\zeta})t - 2i\bar{\zeta}x}}{D(x, t)}, \\ r(x, t) &= -\frac{2ic\, e^{-2A_0(\zeta)t + 2i\zeta x}}{D(x, t)}, \\ \int_x^\infty q(x, t)r(x, t)\, dx &= \frac{2ic\bar{c}\, e^{2(A_0(\bar{\zeta}) - A_0(\zeta))t + 2i(\zeta - \bar{\zeta})x}}{(\zeta - \bar{\zeta})D(x, t)}, \end{aligned} \tag{5.3}$$

where

$$D(x, t) = 1 - \frac{c\bar{c}}{(\zeta - \bar{\zeta})^2}\, e^{2(A_0(\bar{\zeta}) - A_0(\zeta))t + 2i(\zeta - \bar{\zeta})x}.$$

The problem is that if $A_0(\zeta)$, $q(x, 0)$, $r(x, 0)$ are unrestricted, $D(x, t) = 0$ at a countable set of points $(x, 0)$. At these points, the homogeneous integral equations corresponding to (5.1) have infinitely many solutions, and (5.1) has no solution. By hypothesis, none of these points (x, t) occurs at $t = 0$, since $q(x, t)$, $r(x, 0)$ are smooth and decay rapidly as $|x| \to \infty$. After a finite time, however, both $q(x, t)$ and $r(x, t)$ become unbounded at a particular location x, when $D(x, t)$ vanishes. Thus, it is possible for q, r to satisfy (4.11) initially, and to evolve according to (1.16) into functions which "burst" at a particular location after a finite time. These bursting solitons cannot occur in the KdV equation, and their existence represents a major difference between evolution equations solved by (1.6) and those solved by (1.14). In that problem, the constraint on the initial data which corresponds to (4.11) is

$$\int_{-\infty}^{\infty} (1 + |x|)|u|\, dx < \infty,$$

but then the solution satisfies this constraint at any future time [Zakharov and Faddeev, 1972]. Moreover the positive definite "energy", $\int_{-\infty}^{\infty} u^2\, dx$, being a constant of the motion, excludes the development of singularities like those in (5.3). The implications of these bursts in a physically derived evolution equation must depend on the problem, and on the limitations of the particular derivation. Their occurrence in a physical problem ordinarily would require a reexamination of the assumptions underlying the model. We note that similar behavior has been observed recently in a problem of resonant wave interaction [Zakharov and Manakov, 1973].

With the additional stipulation

$$r(x, t) = \alpha q^*(x, t) \quad (\alpha \text{ real, constant}), \tag{5.4}$$

this situation never arises. The first conserved density, $\int_{-\infty}^{\infty} qr\, dx$, is time-invariant and bounds $\int_{x}^{\infty} qr\, dx$. It follows that $D(x, t) \neq 0$, and (5.3) provides the global solution of the evolution equation.

We note that (5.4) includes both of the special cases (4.51, 4.52) in which a unique solution of (5.1) is known to exist. As stated in Section IV, conditions which are both necessary and sufficient for a solution of (5.1) to exist have not been derived. In the remainder of this section, we assume that a unique solution of (5.1) exists.

The solution (5.3) can also be written as

$$q(x, t) = i\bar{c}\, \bar{e}^{i\phi} \operatorname{sech} \theta, \tag{5.5}$$

where

$$\phi = i(A_0(\zeta) + A_0(\bar{\zeta}))t + (\zeta + \bar{\zeta})x - i\gamma,$$

$$\theta = (A_0(\zeta) - A_0(\bar{\zeta}))t + i(\zeta - \bar{\zeta})x + \gamma,$$

and

$$e^{2\gamma} = -\frac{c\bar{c}}{(\zeta - \bar{\zeta})^2}.$$

This is the basic soliton solution: a permanent, localized wave which travels with speed

$$V = \operatorname{Re}\left\{\frac{A_0(\bar{\zeta}) - A_0(\zeta)}{-i(\zeta - \bar{\zeta})}\right\}, \tag{5.6}$$

with amplitude proportional to $(\zeta - \bar{\zeta})$, and wavelength to $1/(\zeta - \bar{\zeta})$. These waves are basically nonlinear; they have no counterpart in linear problems. We mention two examples of their forms in specific contexts. In the Zakharov-Shabat problem (1.15), $A_0(\zeta) = -2i\zeta^2$, $r = -q^*$, $\bar{c} = c^*$, $\bar{\zeta} = \zeta^* = \xi - i\eta$ and

$$q(x, t) = 2\eta\, e^{\{-4i(\xi^2 - \eta^2)t - 2i\xi x + i\phi\}} \operatorname{sech}\{2\eta(x - x_0) + 8\eta\xi t\}.$$

Here the soliton is an envelope of oscillating waves. The amplitude and wavelength depend on η, and the envelope travels as a permanent wave with speed 4ξ. In the sine-Gordon equation

$$u_{xt} = \sin u, \tag{5.7}$$

the physical variables are

$$X = x + t, \qquad T = x - t, \qquad u = -\int_{-\infty}^{x} 2q\, dz.$$

Here,

$$A_0(\zeta) = \frac{i}{4\zeta}, \qquad r = -q, \qquad \bar{\zeta} = -\zeta = -i\eta, \qquad \bar{c} = -c,$$

and the solitons,

$$u(X, T) = 4\tan^{-1}\left\{\exp\left[\left(\eta + \frac{1}{4\eta}\right)(X - X_0) + \left(\eta - \frac{1}{4\eta}\right)T\right]\right\}, \tag{5.8}$$

are known as kinks.

For any of the problems solved by (1.14), as long as the spectrum is purely discrete, the integral kernel is degenerate, and (5.1) always can be solved in closed form (if a solution exists). One obtains the N-soliton solution for the evolution equation in this way, which has been computed explicitly for several specific equations [Hirota, 1971; Hirota, 1973; and Gibbon et. al., 1973]. For large time, the solitons with different speeds separate, and the asymptotic solution is N well-separated waves of the form (5.5). This separation process has been discussed in detail for $r = -q^*$, by Zakharov and Shabat [1972]. They show that the asymptotic effect on each soliton of such an interaction is a phase shift, which can be computed as if each soliton interacts with the other $(N - 1)$ solitons pairwise.

From (5.6), it is clear that there is a locus of eigenvalues whose solitons all have the same speed. These solitons do not separate as $t \to \infty$, but instead form a multisoliton structure, a phenomenon that cannot occur in a solution of the KdV equation. In the Zakharov-Shabat problem, the locus is defined by $\mathrm{Re}\,\{\zeta\} = \xi_0$ and was analyzed by them. For the sine-Gordon equation, the locus is given by $|\zeta| = c_0$, and has been discussed by Lamb [1971], and Ablowitz, Kaup, Newell and Segur [1973a]. These solutions demand special attention in the case ($r = -q$, real), because the eigenvalues occur in complex conjugate pairs $(\zeta, -\zeta^*)$. Thus, any discrete eigenvalue for which $\mathrm{Re}\,(\zeta) \neq 0$ generates such a solution. In the context of the sine-Gordon equation, this solution is

$$u(X, T) = 4\tan^{-1}\left[\frac{\eta\cos\{\xi(\eta(T - T_0) - (4 - v)X)\}}{\xi\cosh\{\eta(v(X - X_0) - (4 - v)T)\}}\right], \tag{5.9}$$

where $v = 2 + (1/2|\zeta|^2)$. As a special case, if $|\zeta| = \frac{1}{2}$, this solution has a fixed location where it oscillates in time.

We mention one other aspect of the solution which is associated entirely with the discrete spectrum. As shown in Section III, an arbitrary ratio of entire functions, $A_0(\zeta)$, generates a nonlinear evolution equation that can be solved by this method. If $A_0(\zeta)$ has any poles, it is necessary to restrict the initial data to insure that no eigenvalues in the scattering problem coincide with the poles of the dispersion relation (see Section III). However, even with this proviso, if any eigenvalues are *near* a pole of $A_0(\zeta)$, the corresponding solitons move with extraordinary speed. This can occur in the sine-Gordon equation $(A_0(\zeta) + i/4\zeta)$, where these large (transformed) speeds give physical speeds very near the speed of light ($v = 1$). In other contexts, of course, these high speeds will have other physical interpretations, but their existence always should be significant.

B. The continuous spectrum

Consider next the contribution from the continuous spectrum to the asymptotic solution of the evolution equation. We begin with the simplest possible case, in which the initial data satisfy:

$$R(\infty)Q(\infty) = \int_{-\infty}^{\infty} |r|\,dx \int_{-\infty}^{\infty} |q|\,dx < .817; \tag{5.10a}$$

also

$$R(\infty)Q(\infty) < .383. \tag{5.10b}$$

As shown in Section VI and below, (5.10a) guarantees that there are no discrete eigenvalues; (5.10b) guarantees the validity of the methods used by Ablowitz and Newell [1973]. The scattering data evolves as

$$F(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{a}(k)\, e^{i(kx + 2iA_0(k)t)}\, dk, \tag{5.11a}$$

$$\bar{F}(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{b}}{\bar{a}}(k)\, e^{-i(kx + 2iA_0(k)t)}\, dk. \tag{5.11b}$$

As $t \to \infty$, the dominant wavenumbers in F, $\bar{F}$ at a particular location x are those whose phase is stationary:

$$\frac{x}{t} + 2iA_0'(k) = 0. \tag{5.12}$$

The contribution to F, $\bar{F}$ from these dominant wavenumbers is found by deforming the path of integration into the path of steepest descents. (We assume that $(b/a)(k)$, $(\bar{b}/\bar{a})(k)$ are both analytic in a region around the real axis large enough for these deformations to be effective.) As a specific example, we consider $A_0(\zeta) = -2i\zeta^2$, for which the evolution equations are

$$\begin{aligned} iq_t + q_{xx} - 2(qr)q &= 0, \\ ir_t - r_{xx} + 2(qr)r &= 0. \end{aligned} \tag{5.13}$$

Then (5.12) becomes

$$\frac{x}{t} = -8k_0,$$

and the deformed path passes through (k_0) at an angle $\pi/4$ for F, and $-\pi/4$ for $\bar{F}$. Asymptotically, as $t \to \infty$, (x/t) fixed,

$$\begin{aligned} F(x, t) &= \frac{1}{4\sqrt{\pi t}}\,\frac{b}{a}\left(-\frac{x}{8t}\right) \exp\left[-\frac{i}{16}\left(\frac{x}{t}\right)^2 t + i\frac{\pi}{4}\right] + O(t^{-3/2}), \\ \bar{F}(x, t) &= \frac{1}{4\sqrt{\pi t}}\,\frac{\bar{b}}{\bar{a}}\left(-\frac{x}{8t}\right) \exp\left[\frac{i}{16}\left(\frac{x}{t}\right)^2 t - i\frac{\pi}{4}\right] + O(t^{-3/2}). \end{aligned} \tag{5.14}$$

The integral equations (5.1) can be combined into

$$K_1(x, y; t) - \bar{F}(x + y; t) + \iint_x^\infty K_1(x, z; t)F(z + s; t)\bar{F}(s + y; t)\, dz\, ds = 0, \tag{5.15}$$

with a similar equation for $\bar{K}_2(x, y; t)$. We seek an approximate solution to (5.15) of the form

$$K_1(x, y; t) = \frac{1}{4\sqrt{\pi t}} f(X, Y) \exp\left[\frac{i}{16}(X + Y)^2 t - \frac{i\pi}{4}\right] + \cdots \tag{5.16}$$

with $X = x/t$, $Y = y/t$.

Substitution of (5.14) and (5.16) into (5.15), with evaluation of the integral by stationary phase [Lewis, 1966] shows that

$$f(X, Y) = \frac{\dfrac{\bar{b}}{\bar{a}}\left(-\dfrac{X + Y}{8}\right)}{1 - \alpha\dfrac{\bar{b}}{\bar{a}}\left(-\dfrac{X + Y}{8}\right)\dfrac{b}{a}\left(-\dfrac{X + Y}{8}\right)}, \tag{5.17}$$

where

$$\alpha = \tfrac{1}{2} \text{ if } X \neq Y,$$
$$= \tfrac{1}{4} \text{ if } X = Y.$$

Then, since $q(x, t) = 2K_1(x, x; t)$,

$$q(x, t) \sim -\frac{1}{2\sqrt{\pi t}} \frac{\dfrac{\bar{b}}{\bar{a}}\left(-\dfrac{x}{4t}\right)}{1 - \dfrac{1}{4}\dfrac{b}{a}\left(-\dfrac{x}{4t}\right)\dfrac{\bar{b}}{\bar{a}}\left(-\dfrac{x}{4t}\right)} \exp\left[\frac{i}{4}\left(\frac{x}{t}\right)^2 t - \frac{i\pi}{4}\right]. \tag{5.18a}$$

Similarly,

$$r(x, t) \sim \frac{1}{2\sqrt{\pi t}} \frac{\dfrac{b}{a}\left(-\dfrac{x}{4t}\right)}{1 - \dfrac{1}{4}\dfrac{b}{a}\left(-\dfrac{x}{4t}\right)\dfrac{\bar{b}}{\bar{a}}\left(-\dfrac{x}{4t}\right)} \exp\left[-\frac{i}{4}\left(\frac{x}{t}\right)^2 t + \frac{i\pi}{4}\right]. \tag{5.18b}$$

The restriction (5.10b) assures that the denominator in (5.17) does not vanish, as shown in Section VI.

In the KdV equation, the solution corresponding to (5.18) is not a uniformly valid asymptotic approximation, and the similarity solution is needed as well. In this problem, (5.18) is uniformly valid, but we still expect the similarity solution to play a (weaker) role in the asymptotic development of the solution. The similarity solution [cf. Benney and Newell, 1967] is

$$q(x, t) = Q_0 t^{-1/2} \exp\left(\frac{i}{4}\frac{x^2}{t} + 2iQ_0R_0 \log t\right),$$

$$r(x, t) = R_0 t^{-1/2} \exp\left(-\frac{i}{4}\frac{x^2}{t} - 2iQ_0R_0 \log t\right), \tag{5.19}$$

where Q_0, R_0 are constants. (5.19) can be matched to (5.18) if and only if (Q_0R_0) is real. In fact, if $|\mathrm{Im}(Q_0R_0)| > \frac{1}{4}$, one of these functions grows without bound as $t \to \infty$. This behavior reflects an inherent instability in the evolution equation (5.13). In regions where the spatial curvatures (q_{xx}, r_{xx}) are small, (5.13) is approximated by the simplified equations

$$\begin{aligned} iq_t - 2(qr)q &= 0, \\ ir_r + 2(qr)r &= 0. \end{aligned} \tag{5.20}$$

Here it is clear that (qr) is a constant, and that either q or r grows exponentially if $\mathrm{Im}(qr) \neq 0$. As with the discrete spectrum, (5.4) insures that this instability does not occur, and that the solutions are well behaved. Thus the discrete and the continuous spectra each can generate solutions of (5.13) which grow without bound. Both of these instabilities are inherently nonlinear, and both are excluded by (5.4).

Thus, if the initial data satisfy (5.10), the solution of the nonlinear evolution equation (r, q) is approximated by (5.18): a slowly-varying, dispersive wave train whose amplitude decays algebraically. Meanwhile, the solution $(F, \bar{F})$ of the linearized evolution equation is approximated by (5.14). Qualitatively the behavior is similar. One expects that if the initial data are "small" the nonlinear terms in the evolution equation are "unimportant", and the solution will be well approximated by the solution of the linearized problem. These statements can now be made precise, by comparing the solutions of the linear and nonlinear problems. In particular, we note that "small" should be interpreted to mean

$$R(\infty)Q(\infty) = \int_{-\infty}^{\infty} |r|\,dx \int_{-\infty}^{\infty} |q|\,dx \ll 1, \tag{5.21}$$

rather than in any "local" sense.

C. *Arbitrary spectra*

In the introduction, we noted the similarities between the method of inverse scattering and the method of Fourier analysis. If in the initial data, $r(x) = 0$, the spectrum is purely continuous and the method actually is the method of Fourier transforms. Moreover, whenever the spectrum is purely continuous, it appears that the asymptotic solution can be obtained by linear methods. For (5.13), we saw that the asymptotic structure of the solution of the nonlinear problem is modeled (locally in x) by that of the solution of the linearized problem. For the KdV equation, the linearized solution is a valid asymptotic approximation in some regions, but not in others. Even so, it is the first term in a regular perturbation expansion which converges uniformly, in all regions [cf. Segur, 1973]. Thus, the continuous spectrum yields a slowly-varying, dispersive wave train whose amplitude decays algebraically, i.e., a solution which is almost linear. On the other hand, a purely discrete spectrum yields solitons and other permanent waves, which have no linear analogue, and cannot be obtained by ordinary linear methods.

Arbitrary initial conditions will require both continuous and discrete spectra, and the asymptotic solution will be more complicated. In the KdV equation (1.5) or in the modified KdV equation (2.9), the solitons all move to the right while the oscillatory wave train moves to the left. Thus, these components separate in space and there is no asymptotic interaction. In the Zakharov-Shabat problem, they do not separate and the solution requires both components simultaneously. The complete asymptotic evaluation can be simplified by using both (4.38) and (4.39), but the details have not been worked out.

We close this section with a brief discussion of one other aspect of the solution of the problem which has no analogue either in the linear problem or in the KdV equation. For k real, there is no bound on $(b/a)(k)$ or $(\bar{b}/\bar{a})(k)$ in this problem as there is in the Schroedinger equation. Similarly, there is no reason *a priori* to expect that the zeros of $a(\zeta)$, $\bar{a}(\zeta)$ cannot be real. As an example of a real zero of $a(\zeta)$ let $r = \alpha q$, α complex. (4.10) can be summed at $\zeta = 0$:

$$a(0) = \bar{a}(0) = \cosh(\sqrt{\alpha}p),$$

where $p = \int_{-\infty}^{\infty} q(x)\,dx$. Thus, one obtains a discrete eigenvalue at $\zeta = 0$ if

$$\sqrt{\alpha}\int_{-\infty}^{\infty} q(x)\,dx = i\left(\frac{\pi}{2} + n\pi\right). \tag{5.22}$$

There is no square-integrable eigenfunction for any discrete eigenvalue on the real axis. Hence, these are called "improper" eigenvalues. Nevertheless, there will be a contribution to the solution of the evolution equation which differs from the components we have discussed so far. A precise description of this contribution is not yet available, except in the following context. For the sine-Gordon equation, $r = -q = \frac{1}{2}(\partial u/\partial x)$, u real. Here (5.22) becomes

$$\begin{aligned} &u \to 0 \quad \text{as } x \to -\infty \\ &u \to \pi + 2n\pi \quad \text{as } x \to +\infty, \end{aligned} \tag{5.23}$$

which is Lamb's [1971] criterion for a π-pulse (or a $(2n + 1)\,\pi$ pulse). His π-pulse is the similarity solution of the evolution equation. In other contexts, the proper interpretation of a real zero of $a(\zeta)$ is not clear.

VI. Estimates of the discrete spectrum

In practical terms, the outstanding feature of problems that can be solved by the inverse scattering method is that their solutions achieve comparatively simple asymptotic states as $t \to \pm\infty$. The contribution from the continuous spectrum decays, and the dominant asymptotic solution is determined by the discrete spectrum of the scattering problem at $t = 0$. In this section, therefore, we derive some simple bounds for the discrete eigenvalues of (1.14); i.e., for the location of the zeros of $a(\zeta)$ and $\bar{a}(\zeta)$.

We derive the results in terms of $a(\zeta)$. If $r(x)$, $q(x)$ are related, the zeros of $\bar{a}(\zeta)$ can be deduced from the zeros of $a(\zeta)$. For $r(x)$, $q(x)$ independent, the computations must be repeated for $\bar{a}(\zeta)$. Further, we assume throughout that

$$\begin{aligned} &\int_{-\infty}^{\infty} |x|^n |q(x)|\,dx < \infty, \\ &\int_{-\infty}^{\infty} |x|^n |r(x)|\,dx < \infty, \end{aligned} \tag{6.1}$$

for all n, so that $a(\zeta)$, $\bar{a}(\zeta)$ are analytic on the real axis, $\operatorname{Im}\zeta = 0$. For brevity, we use

$$R = \int_{-\infty}^{\infty} |r|\,dx, \qquad Q = \int_{-\infty}^{\infty} |q|\,dx. \tag{6.2}$$

1. $a(\zeta)$ has only finitely many zeros in $\operatorname{Im}(\zeta) \geq 0$. As shown in Section IV, (6.1) guarantees that $a(\zeta)$ is analytic for $\operatorname{Im}(\zeta) \geq 0$ and $a(\zeta) \to 1$ as $|\zeta| \to \infty$. It follows that the zeros of $a(\zeta)$ are all isolated, and lie in a bounded region. Hence, $a(\zeta)$ can have at most a finite number of zeros there.

2. $a(\zeta)$ can have zeros on $\operatorname{Im}(\zeta) = 0$, but these are not proper eigenvalues, because they have no square-integrable eigenfunction. As noted in Section V, solutions of the evolution equation that correspond to proper and improper discrete eigenvalues are qualitatively different. Solitons require $\operatorname{Im}(\zeta) \neq 0$.

3. Let N denote the number of zeros of $a(\zeta)$ with $\operatorname{Im}(\zeta) > 0$, with the multiplicity of non-simple zeros included. Let $|\zeta_0|$ denote the radius of a circle which contains all the zeros of $a(\zeta)$, and let $\xi_+ > |\zeta_0|$, $\xi_- < -|\zeta_0|$. Then as $|\xi_\pm| \to \infty$,

$$\frac{1}{2\pi}\{\arg(a(\xi_+)) - \arg(a(\xi_-))\} \to N. \tag{6.3}$$

This can be seen by integrating $a'(\zeta)/a(\zeta)$ around a closed contour C that encloses all the proper discrete eigenvalues in the upper $\frac{1}{2}$-plane, but excludes those on the real axis. $a(\zeta)$ is analytic inside and on C, so

$$\oint_C \frac{a'(\zeta)}{a(\zeta)}\,d\zeta = 2\pi i N.$$

But for large $|\zeta|$, $a(\zeta) \to 1$ and there is no contribution to the integral. For real ζ, $\log(a(\zeta)) \to i\arg(a(\xi_\pm))$ as $\zeta \to \xi_\pm$, from which (6.2) follows. We estimate $|\zeta_0|$ below.

4. As shown in Section IV, if r is proportional to q or to q^*, the zeros of $\bar{a}(\zeta)$ are paired with the zeros of $a(\zeta)$. If, in addition, $q(x)$ and $r(x)$ are real, the zeros of $a(\zeta)$ itself occur in pairs. As shown in Section V, this pair of eigenvalues is associated with a special solution of the evolution equation (called breather, 0π pulse or bion) of a different character than the usual soliton solution.

5. If $r(x) = +q^*(x)$, the eigenvalue problem (1.14) is formally self-adjoint. There are no eigenvalues with $\operatorname{Im}(\zeta) > 0$.

6. For r, q arbitrary, $a(\zeta)$ has no zeros for $\operatorname{Im}(\zeta) \geq 0$ if

$$RQ = \int_{-\infty}^{\infty} |r|\,dx \int_{-\infty}^{\infty} |q|\,dx < .817, \tag{6.4a}$$

or more precisely if

$$I_0(2\sqrt{RQ}) < 2. \tag{6.4b}$$

We show that (6.4) implies that for $\operatorname{Im}(\zeta) \geq 0$,

$$|a(\zeta) - 1| < 1.$$

From (4.6), and (4.9) and

$$\lim_{x\to\infty} \phi_1(x)\, e^{i\zeta x} = a(\zeta),$$

$$a(\zeta) - 1 = \int_{-\infty}^{\infty} r(z) \int_z^{\infty} q(z)\, e^{2i\zeta(y-z)}\, dy(\phi_1\, e^{i\zeta z})\, dz$$

$$|a(\zeta) - 1| \le \int_{-\infty}^{\infty} |r(z)| \int_z^{\infty} |q(y)|\, dy |\phi_1\, e^{i\zeta z}|\, dz \le I_0(2\sqrt{RQ}) - 1. \tag{6.5}$$

Hence, we need

$$I_0(2\sqrt{RQ}) < 2,$$

which is (6.4).

7. Asymptotic methods used in Section V require not only (6.4), but also

$$\left|\frac{b\bar{b}}{a\bar{a}}(\zeta)\right| < 2 \tag{6.6}$$

for all real ξ. Because

$$a\bar{a} + b\bar{b} = 1,$$

this can be rewritten as

$$|1 - a\bar{a}(\xi)| < 2|a\bar{a}(\xi)|.$$

If we let $a\bar{a}(\xi) = \alpha + i\beta$, this becomes

$$\left(\alpha + \frac{1}{3}\right)^2 + \beta^2 > \left(\frac{2}{3}\right)^2.$$

Thus we require $|a\bar{a}| > \frac{1}{3}$, and

$$|a| > \frac{1}{\sqrt{3}}, \qquad |\bar{a}| > \frac{1}{\sqrt{3}}.$$

Finally, we require

$$|a(\xi) - 1| < 1 - \frac{1}{\sqrt{3}},$$

$$|\bar{a}(\xi) - 1| < 1 - \frac{1}{\sqrt{3}},$$

and observe that both are satisfied for all real ξ if

$$I_0(2\sqrt{RQ}) < 2 - \frac{1}{\sqrt{3}}, \tag{6.7}$$

i.e.,

$$RQ < .383.$$

which is (5.10b).

8. Approximations of the largest (in magnitude) eigenvalues (ζ_0) can be obtained in a variety of ways, all of which require some additional smoothness of

$q(x)$, $r(x)$. We derive first three upper bounds on $|\zeta_0|$, and conclude this section with two approximate methods to estimate ζ_0 explicitly.

(a) Let $q(x)$ be continuously differentiable for all x and let

$$q'_m = \max_x |q'(x)|, \tag{6.8}$$

$$A = \int_{-\infty}^{\infty} |qr|\, dx,$$

and

$$B = I_0(2\sqrt{RQ}).$$

If

$$|\zeta| > \frac{B}{4}\left[A + \left\{A^2 + \frac{4Rq'_m}{B}\right\}^{1/2}\right], \tag{6.9}$$

then

$$a(\zeta) \neq 0.$$

Thus, all the discrete eigenvalues must lie within a circle whose radius is given by (6.9).

As above, we show that (6.9) implies

$$|a(\zeta) - 1| < 1.$$

From (6.5)

$$\begin{aligned} I &= \int_z^{\infty} q(y)\, e^{2i\zeta(y-z)}\, dy = \int_0^{\infty} q(z+p)\, e^{2i\zeta p}\, dp \\ &= q(z)\int_0^{\infty} e^{2i\zeta p}\, dp + \int_0^{\infty} q'(z+m)p\, e^{2i\zeta p}\, dp, \end{aligned}$$

where $0 < m < p$.

$$|I| \leq \frac{|q(z)|}{2|\zeta|} + \frac{q'_m}{4\eta^2}, \tag{6.10}$$

where $\eta = \operatorname{Im}(\zeta)$. Using (6.10) and (4.9) in (6.5) we require

$$I_0(2\sqrt{RQ})\left\{\frac{1}{2|\zeta|}\int_{-\infty}^{\infty} |rq|\, dx + \frac{1}{4\eta^2}R_{\infty}q'_m\right\} < 1.$$

Using $|\zeta|^2 \geq \eta^2$ and (6.8) yields

$$|\zeta|^2 > B\left\{\frac{A|\zeta|}{2} + \frac{Rq'_m}{4}\right\},$$

from which (6.9) follows.

(b) This procedure can be carried further to obtain a better bound than (6.9). To illustrate, let $q(x)$ be continuously twice-differentiable, let

$$q''_m = \max_x |q''(x)|,$$

and let ζ_1 satisfy

$$I_0(2\sqrt{RQ})\left\{\frac{1}{2|\zeta_1|}\int_{-\infty}^{\infty}|rq|\,dx+\frac{1}{4|\zeta_1|^2}\int_{-\infty}^{\infty}|rq'|\,dx+\frac{1}{8\eta_1^3}Rq_m''\right\}<1. \tag{6.11}$$

Then $a(\zeta) \neq 0$ if $|\zeta| > |\zeta_1|$. The procedure is identical.

(c) Note that the quantities obtained in (6.9) and (6.11) are related to the polynomial conserved densities of integral rank [Miura, Gardner and Kruskal, 1968]. Following Zakharov and Shabat [1972] another bound on $|\zeta_0|$ can be obtained which involves the conservation laws directly. If for some $\zeta_0 > 0$,

$$\sum\frac{|c_n|}{|2\zeta_0|^n}<\infty, \tag{6.12}$$

then $a(\zeta) \neq 0$ for $|\zeta| > \zeta_0$.

(d) In addition to these bounds on $|\zeta_0|$, approximate estimates of ζ_0 can also be found. The simplest estimate is obtained from the asymptotic expansions of $a(\zeta)$, given by (4.18). If

$$\left|\frac{q'}{q}\right|\ll\left|\int_{-\infty}^{\infty}rq\,dx\right|, \tag{6.13}$$

the $O(\zeta^{-2})$ term can be neglected in that expansion and

$$2i\zeta_0 \doteq \int_{-\infty}^{\infty}rq\,dx. \tag{6.14}$$

Here we have implicitly assumed that $\mathrm{Re}\{\int_{-\infty}^{\infty}rq\,dx\} < 0$, so that ζ_0 lies in the upper $\frac{1}{2}$-plane. If

$$\mathrm{Re}\left\{\int_{-\infty}^{\infty}rq\,dx\right\}>0, \tag{6.15}$$

the result suggests that there are no proper discrete eigenvalues. In the case $r = -1$, when (1.14) reduces to the Schroedinger equation (1.6), (6.14) reduces to the approximation of the first eigenvalue derived by Landau and Lifshitz [1965, p. 156].

(e) A finite number of the conservation laws can be used to approximate the discrete spectrum if one neglects the contribution from the continuous spectrum. This procedure has been used by Karpman [1967, 1968], Lamb [1971] and Kruskal [1974] in particular problems. Karpman [1968] discussed the validity of this procedure for the KdV equation.

VII. Transformations and conservation laws

The analysis in the first six sections of this paper was mainly concerned with finding, classifying and solving those evolution equations which could be solved by the inverse scattering transform method using (2.1) as the linear eigenvalue problem. The "inverse" question as to how, for a given evolution equation, the approximate eigenvalue problem is chosen is another matter and as yet not fully resolved. However, in recent months, many authors (Lamb [1974], McLaughlin and Scott [1973], Wadati [1974], Wahlquist and Estabrook [1974] and ourselves) have made some progress in showing the interrelation between the choice of eigenvalue problem and a class of nonlinear transformations (called Backlund transformations) relating solutions of the given evolution equation. The idea is that if one can find the Backlund transformation (for a given problem) one can also find the appropriate eigenvalue problem.

Indeed it was just such a transformation which led Gardner, Greene, Kruskal and Miura [1967] to the Schroedinger equation. Miura [1968] had found that the transformation

$$u(x, t) = q^2(x, t) - iq_x(x, t), \tag{7.1}$$

related solutions of the equations

$$u_t + 6uu_x + u_{xxx} = 0, \tag{7.2}$$

and

$$q_t + 6q^2q_x + q_{xxx} = 0. \tag{7.3}$$

Treating (7.1) as a Ricatti equation for q, using the standard linearization procedure, and adding an arbitrary constant to $u(x, t)$ to utilize the Galilean invariance property of (7.2), yields the Schroedinger equations with potential $u(x, t)$. Similarly, it was a transformation suggested by Kruskal [1974] between solutions of

$$u_{xt} = \sin u, \tag{7.4}$$

and

$$\frac{v_{xt}}{\sqrt{1 - \varepsilon^2 v^2}} = \sin v, \tag{7.5}$$

which first led us to the correct choice of scattering problem for treating the sine-Gordon equation. At the same time, Lamb [1974] had used the already known transformation between solutions of (7.4) to arrive at the same eigenvalue problem in Schroedinger form. In fact, if we set $r = -q$ in (2.1), we can easily show that

$$(v_1 - iv_2)_{xx} + (\zeta^2 + q^2 - iq_x)(v_1 - iv_2) = 0, \tag{7.6}$$

which suggests that the transformation (7.1) relates solutions of the KdV equation with all those equations which we have listed, consistent with the choice $r = -q$. (The transformations may not be (1:1).)

Backlund transformations were originally defined as transformations between the solutions of second order partial differential equations [Forsythe, 1959]. Consider the relations

$$F_k(\mathbf{x}, \mathbf{p}, u; \mathbf{x}', \mathbf{p}', u') = 0, \qquad 1 \le k \le \alpha \tag{7.7}$$

where $\mathbf{x}, \mathbf{x}', \mathbf{p}, \mathbf{p}'$ are n-dimensional vectors, u and u' scalars and $\alpha < 2n + 1$. If we now insist that $u(\mathbf{x})$ and $u'(\mathbf{x}')$ are solution surfaces of second order partial differential equations whose gradients are $\mathbf{p}(\mathbf{x})$ and $\mathbf{p}'(\mathbf{x}')$ respectively, the set of relations (7.7) constitute what we shall call a Backlund transformation. If the scalars $u(\mathbf{x})$, $u'(\mathbf{x}')$ satisfy higher order partial differential equations, the appropriate higher order derivatives should be included. Examples of such cases are listed below.

In a number of cases (the sine-Gordon, the sinh-Gordon, the KdV and modified KdV equations) there is a direct link between (i) the dual pair of equations (2.1) and (2.2) (the linear eigenvalue problem and associated time dependent equations) and (ii) Backlund transformations relating solutions of evolution equations which are integrability conditions of (2.1), (2.2).

The first case we consider is the sine-Gordon for which equations (2.1) and (2.2) are:

$$
\begin{aligned}
v_{1x} + i\zeta v_1 &= -\frac{u_x}{2} v_2, \\
v_{2x} - i\zeta v_2 &= \frac{u_x}{2} v_1, \\
v_{1t} &= \frac{i}{4\zeta}\cos u v_1 + \frac{i}{4\zeta}\sin u v_2, \\
v_{2t} &= \frac{i}{4\zeta}\sin u v_1 - \frac{i}{4\zeta}\cos u v_2.
\end{aligned} \tag{7.8}
$$

The integrability condition (cross differentiation) on (7.8) implies that u satisfies the sine-Gordon equation

$$u_{xt} = \sin u. \tag{7.9}$$

Define $\gamma = v_2/v_1$, and find the following pair of Ricatti equations for γ:

$$\gamma_x = 2i\zeta\gamma + \frac{u_x}{2}(1 + \gamma^2), \tag{7.10}$$

$$\gamma_t = -\frac{i}{2\zeta}\cos u\gamma + \frac{i}{4\zeta}\sin u(1 - \gamma^2). \tag{7.11}$$

Define

$$\gamma = \tan\frac{u + v}{4}$$

and find,

$$v_x - u_x = 4i\zeta \sin\frac{u + v}{2}, \tag{7.12}$$

$$v_t + u_t = \frac{i}{\zeta}\sin\frac{u - v}{2}. \tag{7.13}$$

If v also satisfies (7.9) then (7.12, 13) is a Backlund transformation relating solutions of the sine-Gordon equation. Rewriting (7.12), (7.13) as

$$u_x - v_x = 4i(-\zeta)\sin\frac{v + u}{2},$$

$$u_t + v_t = \frac{i}{(-\zeta)}\sin\frac{v - u}{2},$$

we can retrace our steps and find that $\gamma = \tan(v + u)/4$ satisfies

$$\gamma_x = 2i(-\zeta)\gamma + \frac{v_x}{2}(1 + \gamma^2),$$

$$\gamma_t = -\frac{i}{2(-\zeta)}\cos v\gamma + \frac{i}{4(-\zeta)}\sin v(1 - \gamma^2),$$

which is equivalent to (7.8) with u replaced by v and ζ by $-\zeta$. But the integrability condition on these two sets of equations requires that

$$v_{xt} = \sin v,$$

and thus (7.12), (7.13) is a Backlund transformation relating solutions of (7.9). The fact that v satisfies (7.9) can be also established directly by cross differentiation of (7.12), (7.13).

By similar reasoning we can show that the Backlund transformation relating two solutions u and v of the sinh-Gordon equation

$$u_{xt} = \sinh u, \tag{7.14}$$

is given by

$$v_x - u_x = 4i\zeta \sinh\frac{u + v}{2},$$

$$v_t + u_t = \frac{i}{\zeta}\sinh\frac{u - v}{2}. \tag{7.15}$$

The Backlund transformation for the modified KdV equation

$$q_t + 6q^2 q_x + q_{xxx} = 0, \tag{7.16}$$

requires more manipulation. Define $q = -u_x/2$ and let p, which also satisfies (7.16) be $-v_x/2$. Then

$$v_x - u_x = 4i\zeta \sin\frac{u + v}{2}, \tag{7.17}$$

$$\frac{v_t + u_t}{4} = (4i\zeta^3 - 2iq^2\zeta)\sin\frac{u + v}{2} + 2iq_x\zeta\cos\frac{u + v}{2} + q_{xx} - 4q\zeta^2 + 2q^3. \tag{7.18}$$

Using (7.17), (7.18) can also be written

$$\frac{v_t + u_t}{4} = (4i(-\zeta)^3 - 2ip^2(-\zeta))\sin\frac{v + u}{2}$$

$$+ 2ip_x(-\zeta)\cos\frac{u + v}{2} + p_{xx} - 4p(-\zeta)^2 + 2p^3, \tag{7.19}$$

and also can be written symmetrically. Using (7.19) and defining

$$\gamma = \tan\frac{u + v}{4} = \frac{v_2}{v_1},$$

we obtain the equation sets,

$$v_{1x} + i(-\zeta)v_1 = +pv_2,$$
$$v_{2x} - i(-\zeta)v_2 = -pv_1,$$
$$v_{1t} = (-4i(-\zeta)^3 + 2ip^2(-\zeta))v_1 + (4p(-\zeta)^2 + 2ip_x(-\zeta) - p_{xx} - 2p^3)v_2,$$
$$v_{2t} = (-4p(-\zeta)^2 + 2ip_x(-\zeta) + p_{xx} + 2p^3)v_1 - (-4i(-\zeta)^3 + 2ip^2(-\zeta))v_2,$$

the integrability condition for which implies p satisfies (7.16).

The Backlund transformation relating solutions of the KdV equation,

$$q_t + 6qq_x + q_{xxx} = 0, \tag{7.20}$$

can be derived with some small modifications. For the KdV equation, (2.1) and (2.2) are:

$$v_{1x} + i\zeta v_1 = qv_2,$$
$$v_{2x} - i\zeta v_2 = -v_1,$$
$$v_{1t} = (-4i\zeta^3 + 2iq - q_x)v_1 + (4\zeta^2 q + 2iq_x\zeta - q_{xx} - 2q^2)v_2,$$
$$v_{2t} = (-4\zeta^2 + 2q)v_1 - (-4i\zeta^3 + 2iq\zeta - q_x)v_2. \tag{7.21}$$

We note that u, where $q = -u_x$ satisfies

$$u_t - 3u_x^2 + u_{xxx} = 0. \tag{7.22}$$

Define $\gamma = v_1/v_2$ and obtain the pair of Ricatti equations

$$\gamma_x = -2i\zeta\gamma + q + \gamma^2 = (\gamma - i\zeta)^2 + q + \zeta^2, \tag{7.23}$$
$$\gamma_t = (-8i\zeta^3 + 4iq\zeta - 2q_x)\gamma + (4\zeta^2 q + 2iq_x\zeta - q_{xx} - 2q^2) - (2q - 4\zeta^2)\gamma^2. \tag{7.24}$$

Let

$$\gamma - i\zeta = \frac{u' - u}{2}$$

and we find from (7.23) that

$$\frac{u'_x + u_x}{2} = \left(\frac{u' - u}{2}\right)^2 + \zeta^2. \tag{7.25}$$

Using (7.22) and (7.25) we find for (7.24)

$$\frac{u'_t - u_t}{2} = -2q_x\left(\frac{u' - u}{2}\right) - 2q\left(\frac{u' - u}{2}\right)^2 + q^2 - 2\zeta^2 q'. \tag{7.26}$$

When $q' = -u'_x$. Using (7.25), equation (7.26) can also be written

$$\frac{u'_x + u_x}{2} = -2q'_x\left(\frac{u - u'}{2}\right) - 2q'\left(\frac{u - u'}{2}\right)^2 + q'^2 - 2\zeta^2 q, \tag{7.27}$$

or symmetrically by taking the average of (7.26), (7.27). Returning steps again shows that u' satisfies (7.22).

While Backlund transformations play many useful roles perhaps their most important feature is that they provide a rational means for finding eigenvalue problems to associate with certain evolution equations. For this reason, it is important to find an independent means for constructing them. There is such a procedure due to Clairin [1903] which, although laborious, has been successfully exploited by Lamb [1974a], McLaughlin and Scott [1974] to obtain Backlund transformations for the sine-Gordon, KdV, modified KdV and nonlinear Schroedinger equations. More recently, Wahlquist and Estabrook [1974] have developed a more sophisticated approach which seems to define more clearly the underlying algebraic structures.

There are other valuable uses which these transformations can serve. For example, an infinite sequence of conservation laws and motion invariants, for the KdV equation, can be generated by the Backlund transformation (to be more precise, one half of the Backlund transformation)

$$u = w + i\varepsilon w_x + \varepsilon^2 w^2/6, \tag{7.28}$$

relating solutions of (7.2) to solutions of the equation

$$w_t + \left(\frac{1}{2}w^2 + \frac{\varepsilon^2}{18}w^2 + w_{xx}\right)_x = 0, \tag{7.29}$$

which has the readily identifiable motion invariant

$$\int w\,dx. \tag{7.30}$$

Solving (7.28) recursively for w as a functional of u in ascending powers of ε and substituting into (7.29) and (7.30) yields the infinite sequence of conservation laws and motion invariants for the KdV equation. Only the even powers of ε contribute.

A similar procedure applied to the Backlund transformation (7.12), treating ζ large, relating solutions of the same equation, does not, contrary to popular belief, give similar results. In fact, the motion invariants transform in a (1:1) fashion

$$\int v_x^2 \to \int u_x^2, \quad \int (-v_{xx}^2 + \tfrac{1}{4}v_x^4) \to \int -u_{xx}^2 + \tfrac{1}{4}u_x^4 \text{ etc.}$$

This observation may suggest a simpler and more direct means of finding these transformations.

The conservation laws have played an important role in the development of the inverse scattering method and in the understanding of a certain class of nonlinear processes. We will now show that the conservation laws and motion invariants for the class of evolution equations discussed in this paper follow directly from (2.3). We may write formally,

$$B = \frac{1}{2i\zeta}(q_t - 2Aq - B_x), \qquad C = \frac{1}{2i\zeta}(-r_t - 2Ar + C_x). \tag{7.31}$$

Substituting (7.31) into the first of the equations in (2.3) and integrating by parts, we find,

$$A_x = -\frac{1}{2i\zeta}(qr)_t + \frac{1}{2i\zeta}(qC + rB)_x - \frac{1}{2i\zeta}(q_xC + r_xB). \tag{7.32}$$

Again, substitution of (7.31) into (7.32) and integrating by parts yields,

$$A_x = -\frac{1}{2i\zeta}(qr)_t + \frac{1}{2i\zeta}(qC + rB)_x - \frac{1}{2i\zeta}(qr_t)_x + \frac{1}{2i\zeta}(2Aqr)_x$$
$$+ (-q_{xx} + 2q^2r)C + (r_{xx} - 2qr^2)B. \tag{7.33}$$

Repeating the process and integrating in x between $-\infty$ and $+\infty$ gives

$$A_+ - A_- + \frac{1}{2i\zeta}\frac{\partial}{\partial t}\int qr\,dx + \frac{1}{(2i\zeta)^2}\frac{\partial}{\partial t}\int \tfrac{1}{2}(rq_x - r_xq)\,dx$$
$$+ \frac{1}{(2i\zeta)^3}\frac{\partial}{\partial t}\int (q_xr_x + q^2r^2)\,dx + \cdots. \tag{7.34}$$

We have assumed either that (a) both q and r tend to zero as $x \to \pm\infty$, (b) or that if r approaches a finite value, B approaches zero at these limits. From (7.34) we see that since ζ is arbitrary, we obtain a sequence of globally conserved quantities when $A_+ = A_-$. The first three are

$$C_1 = \int qr\,dx, \qquad C_2 = \int \tfrac{1}{2}(rq_x - r_xq), \qquad C_3 = \int (q_xr_x + q^2r^2)\,dx. \tag{7.35}$$

We note further that a knowledge of specific A, B, C as power series in ζ would enable us to write down the fluxes (F_n) corresponding to the densities appearing as integrals in (7.35) and write a corresponding sequence of local conservation laws

$$\frac{\partial T_n}{\partial t} + \frac{\partial F_n}{\partial x} = 0. \tag{7.36}$$

We remark in particular that the existence of motion invariants is associated with the relation $A_+ = A_-$. From Section III, we note that if $B_+ = C_+ = 0$ (there is a "single" dispersion relation for the coupled equations (1.16) and (3.26)), this condition is equivalent to the consistency (in time) of the transmission coefficients $a(\zeta, t)$ and $\bar{a}(\zeta, t)$.

As a final remark to this section we consider another class of transformations which allow nonlinear equations to be solved by a sequence of linear methods. Suppose $u(x, t)$ is the dependent variable of some evolution equation. Let $v(x, t)$ be defined for each time t as,

$$v_x = u(x, t)v. \tag{7.37}$$

Naturally as $u(x, t)$ evolves in time, we expect $v(x, t)$ will. Let

$$v_t = Av, \tag{7.38}$$

where A is a functional of u (and its derivatives, integrals). The compatibility of (7.37), (7.38) requires

$$u_t = A_x, \tag{7.39}$$

which is our evolution equation which we will attempt to solve using the same philosophy used in IST. Namely, at $t = 0$, map $u(x, 0)$ into $v(x, 0)$ by (7.37); follow the evolution of $v(x, t)$ by (7.38) till some chosen time t and then invert the mapping. However, this procedure can only be followed if the right hand side of (7.38) (i.e., Av) is independent of the unknown $u(x, t)$. For example, if $A = u$, then $v_t = uv = v_x$ and the evolution equation is $u_t = u_x$. If $A = u^2 + u_x$, $v_t = (u^2 + u_x)v = v_{xx}$, the linear diffusion equation, and the evolution equation is $u_t = 2uu_x + u_{xx}$, which is Burgers equation. In fact, this transformation has already been suggested and used in the literature [Cole, 1951]. Moreover, it is fairly clear how to continue and construct evolution equations which give rise to $u_t = \sum_n a_n(\partial^{2n}/\partial x^{2n})v$. Furthermore, matrix equations of the Burgers' type can be generated and solved. Additional space dimensions may also be included. The main point which we wish to stress here is that for this class of equations, the time evolution of the transformed variable $v(x, t)$ must be followed for all x. This necessitates A being such a special function of u, that, when it acts on v, it becomes an operator independent of $u(x, t)$. The power of IST is that only the asymptotic behavior of $v(x, t)$ as $(x \to \pm\infty)$ need be followed. The price one pays, of course, is that one is faced with tougher analysis in which one artificially introduces another dimension characterized by the eigenvalue ζ. It is a price worth paying.

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Appendix 1

One way† to solve the system (2.3) is as follows. Write (2.1) and (2.2) in the form

$$v_x = Mv,$$

$$v_t = Nv, \tag{A1.1}$$

where

$$M = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix},$$

$$N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{A1.2}$$

Cross differentiation yields the equation,

$$M_t - N_x + MN - NM = 0, \tag{A1.3}$$

which is equivalent to (2.3). This is also equivalent to the formulation of Lax [1968], $L_t = [B, L]$. Using (3.1–3.3) the solutions for A, B, C in matrix notation can be found by assuming the form $N = \Phi S \Phi^{-1}$ where Φ is the fundamental matrix

$$\Phi = \begin{pmatrix} \phi_1 & \bar{\phi}_1 \\ \phi_2 & \bar{\phi}_2 \end{pmatrix}, \tag{A1.4}$$

and Φ^{-1} is given by,

$$\Phi^{-1} = \begin{pmatrix} -\bar{\phi}_2 & \bar{\phi}_1 \\ \phi_2 & -\phi_1 \end{pmatrix}. \tag{A1.5}$$

Note in deriving (A1.5) we have used the relation $\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2 = -1$. Substitution of N into (A1.3) yields $M_t = \Phi S_x \Phi^{-1}$ whereupon S is given by the equation

$$S = S(-\infty) + \int_{-\infty}^{x} \Phi^{-1} M_t \Phi \, dx, \tag{A1.6}$$

† This particular formulation was suggested by David Levermore.

As in section 3, as $x \to -\infty$,

$$A \to A_-(\zeta),$$
$$B, C \to 0, \tag{A1.7}$$

hence we find

$$S(-\infty) \equiv S_- = N_- = A_-(\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A1.8}$$

Using $N = \Phi S \Phi^{-1}$ (A1.6) yields the solutions (in matrix form) for A, B, and C. Furthermore as $x \to +\infty$,

$$N_+ = \lim_{x \to +\infty} \begin{pmatrix} A_+ & B_+ e^{-2i\zeta x} \\ C_+ e^{2i\zeta x} & -A_+ \end{pmatrix}, \tag{A1.9}$$

and hence $S_+ = \Phi_+^{-1} N_+ \Phi_+$ may be easily computed. Note that Φ_+ is given by

$$\Phi_+ = \lim_{x \to +\infty} \begin{pmatrix} a\, e^{-i\zeta x} & \bar{b}\, e^{-i\zeta x} \\ b\, e^{i\zeta x} & -\bar{a}\, e^{i\zeta x} \end{pmatrix}. \tag{A1.10}$$

Using (A1.9–A1.10) in (A1.6) yields as $x \to +\infty$ three equations for A_+, B_+, and C_+:

$$A_+(a\bar{a} - b\bar{b}) + C_+ a\bar{b} + B_+ \bar{a} b = A_- + I(\phi, \bar{\phi}),$$
$$2A_+ \bar{a}\bar{b} + C_+ \bar{b}^2 - B_+ \bar{a}^2 = I(\bar{\phi}, \bar{\phi}),$$
$$2A_+ ab - C_+ a^2 + B_+ b^2 = -I(\phi, \phi), \tag{A1.11}$$

where we have used the definition $I(u, v)$ (3.9). Equations (A1.11) can be inverted to find A_+, B_+, C_+ as

$$A_+ = A_-(a\bar{a} - b\bar{b}) + I(\phi, \bar{\phi})(a\bar{a} - b\bar{b}) + abI(\bar{\phi}, \bar{\phi}) - \bar{a}\bar{b}I(\phi, \phi),$$
$$B_+ = 2a\bar{b}A_- + 2a\bar{b}I(\phi, \bar{\phi}) - a^2 I(\bar{\phi}, \bar{\phi}) - \bar{b}^2 I(\phi, \phi),$$
$$C_+ = 2\bar{a}bA_- + 2\bar{a}bI(\bar{\phi}, \phi) + b^2 I(\bar{\phi}, \bar{\phi}) + \bar{a}^2 I(\phi, \phi). \tag{A1.12}$$

Using (3.9–3.10) the above set of equations can be put into the simplified form (3.10) given in Section III.

Appendix 2

In the special case when $r = \pm q$ the operator equation:

$$\sigma_3 u_t + 2A_-(L^+)u = 0,$$

(3.27) may be put in scalar form. When $r = \pm q$, L^+ (3.22) may be written as

$$L^+ = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \tag{A2.1}$$

where

$$\alpha = \frac{1}{2i}\left(\partial/\partial x \mp 2q \int_{-\infty}^{x} dy\, q\right), \quad \beta = \frac{1}{2i}\left(2q \int_{-\infty}^{x} dy\, q\right).$$

Alternatively,

$$L^{+} = \sigma_3 P(\alpha + \beta) + \sigma_3 Q(\alpha - \beta), \tag{A2.2}$$

with P, Q given by

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$Q = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{A2.3}$$

Using the relations

$$\begin{aligned}
\sigma_3 P \sigma_3 Q &= Q = QQ, \\
\sigma_3 P \sigma_3 P &= \sigma_3 Q \sigma_3 Q = QP = PQ = 0, \\
\sigma_3 Q \sigma_3 P &= P = PP, \\
\sigma_3 P + \sigma_3 Q &= \sigma_3,
\end{aligned} \tag{A2.4}$$

we find,

$$\begin{aligned}
(L^{+})^{2n} &= Q[(\alpha + \beta)(\alpha - \beta)]^n + P[(\alpha - \beta)(\alpha + \beta)]^n, \\
(L^{+})^{2n+1} &= \sigma_3 P(\alpha + \beta)[(\alpha - \beta)(\alpha + \beta)]^n + \sigma_3 Q(\alpha - \beta)[(\alpha + \beta)(\alpha - \beta)]^n.
\end{aligned} \tag{A2.5}$$

Writing $A(L^{+})$ in terms of its even and odd parts we can replace the original operator equation by its scalar equivalent. Indeed, it is straightforward to show that we arrive at consistent equations so long as $A_{-}(L^{+})$ is an odd function of its argument. When $A_{-}(L^{+})$ is odd we have,

$$\begin{aligned}
A_{-}(L^{+}) &= \sigma_3 P(\alpha + \beta) f((\alpha - \beta)(\alpha + \beta)) + \sigma_3 Q(\alpha - \beta) f((\alpha + \beta)(\alpha - \beta)) \\
&= \sigma_3 P f((\alpha + \beta)(\alpha - \beta))(\alpha + \beta) + \sigma_3 Q f((\alpha - \beta)(\alpha + \beta))(\alpha - \beta)
\end{aligned} \tag{A.26}$$

where

$$f(\zeta^2) = A_{-}(\zeta)/\zeta. \tag{A2.7}$$

In this limit the evolution equation (3.27) reduces to,

$$q_t + 2f[(\alpha \pm \beta)(\alpha \mp \beta)](\alpha \pm \beta)q = 0, \tag{A2.8}$$

when $r = \pm q$.

Substitution of α, β yield the evolution equations:

$$q_t = if\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2} \pm q_x \int_{-\infty}^{x} dy\, q \pm q^2\right) q_x = 0, \tag{A2.9}$$

when $r = \pm q$.

Finally, it should be noted that the general evolution operator is simply a differential operator in the linear limit since

$$L^{+} \underset{\substack{q \to 0 \\ r \to 0}}{\sim} \frac{1}{2i}\begin{pmatrix} \partial/\partial x & 0 \\ 0 & -\partial/\partial x \end{pmatrix} = \frac{1}{2i}\sigma_3 \frac{\partial}{\partial x}. \tag{A2.10}$$

In the limit of small amplitudes the evolution equation becomes

$$\begin{pmatrix} r \\ q \end{pmatrix}_t + 2\sigma_3 A_-\left(\frac{1}{2i}\sigma_3\frac{\partial}{\partial x}\right)\begin{pmatrix} r \\ q \end{pmatrix} = 0. \tag{A2.11}$$

Thus,

$$r_t + 2A_-\left(\frac{1}{2i}\frac{\partial}{\partial x}\right)r = 0,$$

$$q_t - 2A_-\left(-\frac{1}{2i}\frac{\partial}{\partial x}\right)q = 0. \tag{A2.12}$$

The linearized dispersion relations for r and q are therefore,

$$\omega_r(k) = 2iA_-(-k/2),$$

$$\omega_q(k) = -2iA_-(k/2), \tag{A2.13}$$

where ω_r, ω_q are defined as the dispersion functions for the special solutions

$$r = e^{-i(kx-\omega_r t)},$$

$$q = e^{-i(kx-\omega_q t)}. \tag{A2.14}$$

Note (A2.13) require

$$\omega_q(k) = -\omega_r(-k) = \omega(k), \tag{A2.15}$$

say.

In terms of these dispersion functions the evolution equation can be written as,

$$u_t + i\sigma_3\omega(2L^+)u = 0, \quad \text{where } u = \begin{pmatrix} r \\ q \end{pmatrix}. \tag{A2.16}$$

Further, for the special case where $r = \pm q$ we have $f(\zeta^2) = i\omega(2\zeta)/2\zeta$, hence (A2.9) yield,

$$q_t + C\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2} \pm q_x\int_{-\infty}^{x} dy\, q \pm q^2\right)q_x = 0, \tag{A2.17}$$

when $r = \pm q$, and where $C(\zeta)^2 = \omega(2\zeta)/2\zeta$ in the linearized phase velocity.

Appendix 3

In the case where $r = -1$ and $q \to 0$ as $|x| \to \infty$ (2.1) is equivalent to the Schroedinger eigenvalue problem,

$$v_{2xx} + (\zeta^2 + q)v_2 = 0, \tag{A3.1}$$

Despite some differences, the general evolution equation may be determined in a manner very similar to (3.27).

The appropriate eigenfunctions for (3.1) when $r = -1$ are defined by the boundary conditions,

$$\phi \sim \begin{pmatrix} 2i\zeta \\ 1 \end{pmatrix} e^{-i\zeta x},$$

$$\bar{\phi} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \quad \text{as } x \to -\infty, \tag{A3.2}$$

and define the scattering data by,

$$\phi \sim \begin{pmatrix} 2i\zeta a & e^{-i\zeta x} \\ a\, e^{-i\zeta x} & +b\, e^{i\zeta x} \end{pmatrix},$$

$$\bar{\phi} \sim \begin{pmatrix} 2i\zeta \bar{b} & e^{-i\zeta x} \\ \bar{a}\, e^{i\zeta x} & +\bar{b}\, e^{-i\zeta x} \end{pmatrix} \quad \text{as } x \to +\infty. \tag{A3.3}$$

From the differential equation and the above relations,

$$a\bar{a} - b\bar{b} = 1. \tag{A3.4}$$

Note that the usual transmission and reflection coefficients for the Schroedinger equation are given by

$$T(\zeta) = 1/a(\zeta),$$

$$R(\zeta) = \frac{b}{a}(\zeta),$$

hence

$$T\bar{T} + R\bar{R} = 1, \tag{A3.5}$$

as is expected.

The analysis can be discussed in complete detail. However, we shall only quote the results for the special case whereas $|x| \to \infty$, $A_+ = A_- = i\zeta C_+ = i\zeta C_- = i\zeta C_0(\zeta)$, and $B_+ = B_- = 0$. Using the notation of appendix 1.1 and formulating the fundamental matrix Φ with ϕ, $\bar{\phi}$ defined by (A3.2–3) we find

$$S(-\infty) = S_- = \Phi^{-1}(-\infty)N(-\infty)\Phi(-\infty)$$
$$= +i\zeta C_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A3.6}$$

and

$$S_+ = i\zeta C_0 \begin{pmatrix} a\bar{a} + b\bar{b} & 2\bar{a}\bar{b} \\ 2ab & -(a\bar{a} - b\bar{b}) \end{pmatrix}, \tag{A3.7}$$

(A1.6) yields the solutions for A, B, C and in the limit $x \to +\infty$, in conjunction with (A3.6–7), yields the orthogonality conditions,

$$J(\phi_2, \phi_2) = 4\zeta^2 ab C_0(\zeta),$$
$$J(\bar{\phi}_2, \bar{\phi}_2) = -4\zeta^2 \bar{a}\bar{b} C_0(\zeta),$$
$$J(\phi_2, \bar{\phi}_2) = -4\zeta^2 b\bar{b} C_0(\zeta), \tag{A3.8}$$

where we have defined,

$$J(\phi_2, \bar{\phi}_2) = \int_{-\infty}^{\infty} q_t \phi_2 \bar{\phi}_2 \, dx, \tag{A3.9}$$

and $J(\phi_2, \phi_2)$, $J(\bar{\phi}_2, \bar{\phi}_2)$ analogously. Further, it is straightforward to verify that,

$$4\zeta^2 ab = \int_{-\infty}^{\infty} (\phi_1^2 - 2i\zeta(\phi_1\phi_2))_x \, dx = \int_{-\infty}^{\infty} q_x \phi_2^2 \, dx,$$

$$4\zeta^2 \bar{a}\bar{b} = \int_{-\infty}^{\infty} (\bar{\phi}_1^2 - 2i\zeta\bar{\phi}_1\bar{\phi}_2)_x \, dx = \int_{-\infty}^{\infty} q_x \bar{\phi}_2^2 \, dx,$$

$$4\zeta^2 b\bar{b} = \int_{-\infty}^{\infty} (\phi_1\bar{\phi}_1 - i\zeta(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2))_x \, dx = \int_{-\infty}^{\infty} q_x \phi_2 \bar{\phi}_2 \, dx, \tag{A3.10}$$

(A3.8) and (A3.10) imply the orthogonality conditions

$$\int_{-\infty}^{\infty} (q_t + C_0(\zeta) q_x) \tilde{\Phi}_i \, dx = 0 \qquad i = 1, 2, 3, \tag{A3.11}$$

where

$$\tilde{\Phi}_1 = \phi_2^2, \qquad \tilde{\Phi}_2 = \bar{\phi}_2^2, \qquad \tilde{\Phi}_3 = \phi_2 \bar{\phi}_2.$$

The general evolution operator may be ascertained by going back to the differential equation

$$\phi_{2xx} + (\zeta^2 + q)\phi_2 = 0, \tag{A3.12}$$

with the boundary condition $\phi_2 \sim e^{-i\zeta x}$ as $x \to -\infty$. Multiplying (A3.12) by ϕ_{2x} yields,

$$\frac{\partial}{\partial x}(\phi_{2x}^2) = -\zeta^2 \frac{\partial}{\partial x}(\phi_2^2) - q \frac{\partial}{\partial x} \phi_2^2. \tag{A3.13}$$

Performing the operation $\int_{-\infty}^{x} dy$ on (A.13) gives,

$$\phi_{2x}^2 = -\zeta^2 \phi_2^2 - \int_{-\infty}^{x} dy \, q \frac{\partial(\phi_2^2)}{\partial y}. \tag{A3.14}$$

After multiplying by ϕ_2, however, (A3.12) may be written in the form

$$\zeta^2 \phi_2^2 = -q\phi_2^2 - \phi_2 \phi_{2xx} = -q\phi_2^2 - \frac{\partial}{\partial x}(\phi_2 \phi_{2x}) + \phi_{2x}^2. \tag{A3.15}$$

Judiciously using (A3.14–15) yields the crucial relation

$$2\zeta^2 \phi_2^2 = -q\phi_2^2 - \frac{1}{2} \frac{\partial^2}{\partial x^2}(\phi_2^2) - \int_{-\infty}^{x} dy \, q \frac{\partial \phi_2^2}{\partial y}. \tag{A3.16}$$

One integration by parts results in the operator

$$\zeta^2\phi_2^2 = L_s\phi_2^2,$$

where

$$L_s = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}\int_{-\infty}^{x} dy\, q_y. \tag{A3.17}$$

Requiring C_0 to be a function of ζ^2 (i.e., $C_0(\zeta^2)$ and as before a quotient of entire functions reduces (A3.11) to

$$\int_{-\infty}^{\infty} (q_t + q_x C_0(L_s))\Phi_i\, dx = 0 \qquad i = 1, 2, 3, \tag{A3.18}$$

As earlier the adjoint operator of $C_0(L_s)$, $C_0(L_s^+)$ results in the general evolution equation

$$q_t + C_0(L_s^+)q_x = 0, \tag{A3.19}$$

where L_s^+ is given by

$$L_s^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}q_x\int_x^{\infty} dy. \tag{A3.20}$$

In the linear limit ($q \to 0$), (A3.20) tends to

$$q_t + C_0\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)q_x = 0. \tag{A3.21}$$

For (A3.21) to possess wavelike solutions, setting $q = e^{ikx - i\omega t}$ yields the dispersion equation,

$$\omega/k = C_0((k/2)^2). \tag{A3.22}$$

Thus in terms of the phase velocity of the linearized problem $\omega/k = \hat{C}(k^2)$ the general evolution equation may be written in the form

$$q_t + \hat{C}(4L_s^+)q_x = 0. \tag{A3.23}$$

which is equation (3.33).

Appendix 4. Alternative form of evolution equations whose linearized form has a singular dispersion relation

If $\Omega_2(\zeta) = (\zeta - \hat{\zeta}_1)\cdots(\zeta - \hat{\zeta}_N)$, we may write (3.36) as the system,

$$\begin{aligned} (L^+ - \hat{\zeta}_1)f^{(1)} &= f^{(2)} \\ &\;\vdots \\ (L^+ - \hat{\zeta}_N)f^{(N)} &= -2\Omega_1(L^+)u, \end{aligned} \tag{A4.1}$$

where $\sigma_3 u_t = f^{(1)}$, $f^{(k)} = (f_1^{(K)}, f_2^{(k)})^T$ and we assume the degree of $\Omega_1(\zeta)$ to be at most $(N - 1)$. Consider any one of these equations in component form

$$(L^+ - \hat{\zeta})f = g,\; g = (g_1, g_2)^T. \tag{A4.2}$$

It proves to be convenient to introduce the auxiliary variable

$$f_3 = \int_{-\infty}^{x} (qf_1 - rf_2)\,dx. \tag{A4.3}$$

Then (A4.2) becomes,

$$\begin{aligned} f_{3x} &= qf_1 - rf_2, \\ f_{2x} + 2i\zeta f_2 &= -2ig_2 - 2f_2 q, \\ f_{1x} - 2i\zeta f_1 &= 2ig_1 + 2f_3 r. \end{aligned} \tag{A4.4}$$

which are the A, B, C equations (2.3) with (q_t, r_t) replaced by $(-2ig_2, 2ig_1)$.

Before discussing the solution, we remark that $f_3^{(k)}(+\infty)$, $k = 1, \ldots, N$ is directly related to the fluxes (at ∞) corresponding to the integrated densities $C_1, C_2, \ldots C_k$. Note that

$$\begin{aligned} f_3^{(1)}(+\infty) &= \int_{-\infty}^{\infty} (qf_1^{(1)} - rf_2^{(1)})\,dx \\ &= \frac{\partial}{\partial t} C_1, \end{aligned} \tag{A4.5}$$

$$\begin{aligned} f_3^{(2)}(+\infty) &= \int_{-\infty}^{\infty} (q, -r)^T \cdot f^{(2)}\,dx \\ &= \int_{-\infty}^{\infty} (q, -r)^T \cdot (L^+ - \hat{\zeta}_1) f^{(1)}\,dx \\ &= \int_{-\infty}^{\infty} (L - \hat{\zeta}_1)(q, -r)^T \cdot f^{(1)}\,dx \\ &= -\frac{1}{2i}\frac{\partial}{\partial t} C_2 - \hat{\zeta}_1 \frac{\partial}{\partial t} C_1. \end{aligned} \tag{A4.6}$$

Similarly, $f_3^{(k)}(+\infty)$ involves the time derivatives of the first k integrated densities. It will turn out that in almost all cases these quantities must be zero in order that equation (3.36) can be inverted and solved for a $\sigma_3 u_t$ which approaches zero at $x \to \pm\infty$. This demand does not lead to problem overspecification for, written in the form (3.36), the equation itself does not specify anything about the time behavior of the first N integrated densities. For example, take $N = 1$, $\Omega_1 = 1$. Then (3.36) is, in component form,

$$r_{xt} - 2r \int_{-\infty}^{x} (qr)_t\,dy - 2i\hat{\zeta}_1 r_t = -4ir, \tag{A4.7}$$

$$q_{xt} - 2q \int_{-\infty}^{x} (qr)_t\,dy + 2i\hat{\zeta}_1 q_t = -4iq. \tag{A4.8}$$

Multiplying (A4.7) by $-q$, (A4.8) by r, adding and integrating gives

$$\frac{\partial C_2}{\partial t} + 2i\hat{\zeta}_1 \frac{\partial C_1}{\partial t} = 0. \tag{A4.9}$$

Thus the evolution equation itself does not specify that C_1 is or is not a motion invariant. However, if C_1 is then so are $\{C_k\}_{k=2}^{\infty}$. Similarly, in general, if $C_1, \ldots, C_N$ are motion invariants, then so are $\{C_k\}_{k=N+1}^{\infty}$. It is also unclear how in the above written form, the evolution equations (A4.7), (A4.8) can be solved as an initial value problem. On the other hand, it is clear how, in principle, the solution for the system (A4.1) may be obtained. First, the system is solved for $f^{(1)} = \sigma_3 u_t$ as operator equations in x. The demand that each $f^{(k)}$, $k = 1, \ldots, N$ and in particular that $\sigma_3 u_t$ approach zero as $|x| \to \infty$ helps remove the ambiguity associated with (A4.7), (A4.8) by causing $\partial C_1/\partial t$ (and also A_+, (see (3.10))) to be specified. Once $\sigma_3 u_t$ is obtained, u can be found for all x at the next time step and the "sweep and step" process repeated. In practice, of course, we will obtain the solution for u at later times by the inverse method.

Returning to (A4.4), we write down its general solution:

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} \phi_1\phi_2 & \bar{\phi}_1\bar{\phi}_2 & \frac{1}{2}(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2) \\ -\phi_1^2 & -\bar{\phi}_1^2 & -\phi_1\bar{\phi}_1 \\ \phi_2^2 & \bar{\phi}_2^2 & \phi_2\bar{\phi}_2 \end{pmatrix} \cdot \left[\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} + 2i \begin{pmatrix} \int_{-\infty}^{x} (g_1\bar{\phi}_1^2 + g_2\bar{\phi}_2^2)\,dy \\ \int_{-\infty}^{x} (g_1\phi_1^2 + g_2\phi_2^2)\,dy \\ -2\int_{-\infty}^{x} (g_1\phi_1\bar{\phi}_1 + g_2\phi_2\bar{\phi}_2)\,dy \end{pmatrix} \right]. \tag{A4.10}$$

Let us first consider the simplest case, namely $N = 1$. Suppose

$$\Omega(\zeta) = \frac{i\alpha}{2(\zeta - \hat{\zeta}_1)}, \left(\text{i.e., } \Omega_2(\zeta) = (\zeta - \hat{\zeta}_1), \qquad \Omega_1(\zeta) = \frac{i\alpha}{2} \right).$$

The system (A4.1) becomes

$$(L^+ - \hat{\zeta}_1)f = -i\alpha u, \tag{A4.11}$$

with $f = (r_t, -q_t)^T$ and $f_3 = (\partial/\partial t)\int_{-\infty}^{x} (qr)\,dy$. In this case, a little calculation shows that,

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} -\alpha(1 + \bar{\phi}_1\bar{\phi}_2 + \bar{\phi}_1\phi_2) \\ 2\alpha\phi_1\bar{\phi}_1 \\ -2\alpha\phi_2\bar{\phi}_2 \end{pmatrix} + K_1 \begin{pmatrix} \phi_1\phi_2 \\ -\phi_1^2 \\ \phi_2^2 \end{pmatrix} + K_2 \begin{pmatrix} \bar{\phi}_1\bar{\phi}_2 \\ -\bar{\phi}_1^2 \\ \bar{\phi}_2^2 \end{pmatrix} + K_3 \begin{pmatrix} \frac{1}{2}(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2) \\ -\phi_1\bar{\phi}_1 \\ \phi_2\bar{\phi}_2 \end{pmatrix}, \tag{A4.12}$$

where all quantities which are functions of ζ are evaluated at $\hat{\zeta}_1$. As $x \to -\infty$, we find, using (3.1), (3.2), (3.3),

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + K_1 \begin{pmatrix} 0 \\ -e^{-2i\hat{\zeta}_1 x} \\ 0 \end{pmatrix} + K_2 \begin{pmatrix} 0 \\ 0 \\ e^{2i\hat{\zeta}_1 x} \end{pmatrix} + K_3 \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}. \tag{A4.13}$$

The solution must satisfy the requirements that

$$f_1, f_2 \to 0, \quad |x| \to \infty, \tag{A4.14}$$

and by definition,

$$f_3(-\infty) = 0.$$

We distinguish the various cases, $\hat{\zeta}_1$ real, $\operatorname{Im}\hat{\zeta}_1 > 0$, $\operatorname{Im}\hat{\zeta}_1 < 0$. First, if $\hat{\zeta}_1$ is real, we must have $K_1 = K_2 = K_3 = 0$. Then as $x \to +\infty$,

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} \to \begin{pmatrix} -\alpha(1 - a\bar{a} + b\bar{b}) \\ 2\alpha a\bar{b}\, e^{-2i\hat{\zeta}_1 x} \\ 2\alpha\bar{a}b\, e^{2i\hat{\zeta}_1 x} \end{pmatrix}. \tag{A4.15}$$

From (A4.14), we must have either

(i) $a(\hat{\zeta}_1) = \bar{a}(\hat{\zeta}_1) = 0$

or

(ii) $b(\hat{\zeta}_1) = \bar{b}(\hat{\zeta}_1) = 0.$ (A4.16)

Note the alternative combination $a(\hat{\zeta}_1) = b(\hat{\zeta}_1) = 0$ and $\bar{a}(\hat{\zeta}_1) = \bar{b}(\hat{\zeta}_1) = 0$ would violate $a\bar{a} + b\bar{b} = 1$. In case (i) $f_3(+\infty) = -2\alpha$ which will correspond to a system which is nonconservative. For case (ii) $f_3(+\infty) = 0$ and all the integrated densities $\{C_k\}_{k=1}^N$ are motion invariants. From (3.10) it follows after a little manipulation that $A_+ = A_- = \Omega$ (recall that without loss of generality we have set $A_- = \Omega$). The evolution equation (3.36) becomes

$$\begin{aligned} r_t &= -2\alpha\phi_2\bar{\phi}_2, \\ q_t &= -2\alpha\phi_1\bar{\phi}_1, \end{aligned} \tag{A4.17}$$

together with the auxiliary equations

$$\begin{aligned} \tfrac{1}{2}(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2)_x &= q\phi_2\bar{\phi}_2 + r\phi_1\bar{\phi}_1 \\ (\phi_2\bar{\phi}_2)_x - 2i\hat{\zeta}_1\phi_2\bar{\phi}_2 &= r(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2) \\ (\phi_1\bar{\phi}_1)_x + 2i\hat{\zeta}_1\phi_1\bar{\phi}_1 &= q(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2). \end{aligned} \tag{A4.18}$$

where the quantities ϕ and $\bar{\phi}$ are estimated at $\zeta = \hat{\zeta}_1$. If $r = -q^*$, the equations correspond directly to the Maxwell-Bloch equations [Ablowitz, Kaup, Newell, 1974] and because of this we will refer to equations of type (A4.18) as generalized Bloch equations. Take $\lambda = 2\phi_1\bar{\phi}_1$ and $N = \phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2$ and we may write,

$$\begin{aligned} q_t &= -\alpha\lambda, \\ \lambda_x + 2i\hat{\zeta}_1\lambda &= 2qN, \\ N_x &= -(q\lambda^* + q^*\lambda), \end{aligned} \tag{A4.19}$$

with the boundary conditions

$$N \to -1, \lambda \to 0 \quad \text{as } x \to -\infty.$$

The fact that $b(\hat{\zeta}_1) = \bar{b}(\hat{\zeta}_1) = 0$ implies the same values are assumed at $x = +\infty$. These equations describe the interaction of the envelope of an electromagnetic wave pulse with an active, resonant, two level medium with polarization λ, density of excited atoms N and detuning parameter $\hat{\zeta}_1$. In particular, if $\hat{\zeta}_1 = 0$ and α, q, λ are real, we may set $q = (-u_x/2)$, $\lambda = \sin u$, $N = -\cos u$, and (A4.19) reduces to the sine-Gordon equation

$$u_{xt} = 2\alpha \sin u. \tag{A4.20}$$

The reader may verify from (2.1) that when $r = -q = u_x/2$ and $\zeta = 0$,

$$\begin{pmatrix} \phi_1(x, t, 0) \\ \phi_2(x, t, 0) \end{pmatrix} = \begin{pmatrix} \cos u/2 \\ \sin u/2 \end{pmatrix} \to \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} \quad \text{as } x \to +\infty. \tag{A4.21}$$

Thus, if $u \to 2n\pi$ as $x \to +\infty$, $b(0)$ (and $\bar{b}(0)$) are zero as required. On the other hand if $u \to (2n+1)\pi$ as $x \to +\infty$, $a(0) = \bar{a}(0) = 0$. Therefore case (i) may very well have physical reality in connection with a special class of boundary conditions. However, the analysis is difficult and will be discussed in a later paper.

We now turn to the case where $\operatorname{Im} \hat{\zeta}_1 > 0$. From (A4.13), we require $K_2 = K_3 = 0$ but not necessarily K_1. As $x \to +\infty$,

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} \to \begin{pmatrix} -\alpha(1 - a\bar{a} + b\bar{b}) + K_1 ab \\ (2\alpha a\bar{b} - K_1 a^2)\, e^{-2i\hat{\zeta}_1 x} \\ (2\alpha \bar{a} b + K_1 b^2)\, e^{2i\hat{\zeta}_1 x} \end{pmatrix}, \tag{A4.22}$$

where all functions of ζ are estimated at $\hat{\zeta}_1$. In order to satisfy (A4.14) we must have either (i) $a(\hat{\zeta}_1) = 0$ or (ii) $K_1 = 2\bar{b}/a$. Case (i) is unacceptable as K_1 is then indeterminate and the solution nonunique. In case (ii), it can be seen that $f_3(+\infty) = 0$ which implies $A_+ = A_- = \Omega$ and leads to the existence of an infinite sequence of motion invariants. The evolution equations becomes, using (3.11)

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = \frac{2\alpha}{a} \begin{pmatrix} \phi_2 \psi_2 \\ \phi_1 \psi_1 \end{pmatrix}, \tag{A4.23}$$

together with the generalized Bloch equations,

$$\begin{aligned} \tfrac{1}{2}(\phi_1 \psi_2 + \phi_2 \psi_1)_x &= q \phi_2 \psi_2 + r \phi_1 \psi_1 \\ (\phi_1 \psi_1)_x + 2i\hat{\zeta}_1 \phi_1 \psi_1 &= q(\phi_1 \psi_2 + \phi_2 \psi_1) \\ (\phi_2 \psi_2)_x - 2i\hat{\zeta}_1 \phi_2 \psi_2 &= r(\phi_1 \psi_2 + \phi_2 \psi_1), \end{aligned} \tag{A4.24}$$

and all functions of ζ are estimated at $\hat{\zeta}_1$. Note that all quantities are defined for $\operatorname{Im} \hat{\zeta}_1 > 0$ (see 4.11).

If $\operatorname{Im} \hat{\zeta}_1 < 0$ a similar situation prevails and one finds

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = -\frac{2\alpha}{\bar{a}} \begin{pmatrix} \bar{\phi}_2 \bar{\psi}_2 \\ \bar{\phi}_1 \bar{\psi}_1 \end{pmatrix}, \tag{A4.25}$$

together with (A4.24) with ϕ, ψ replaced by $\bar{\phi}, \bar{\psi}$.

In summary then: if $\hat{\zeta}_1$ is real, we must demand $b(\hat{\zeta}_1) = \bar{b}(\hat{\zeta}_1) = 0$. If $\operatorname{Im} \hat{\zeta}_1 \gtrless 0$, equation (3.36) may be inverted without further specification save for the demand that $a(\hat{\zeta}_1)(\bar{a}(\hat{\zeta}_1))$ is nonzero. In all cases, an infinite sequence of conservation laws exists. When $\Omega_2(\zeta)$ is a finite product of factors $(\zeta - \hat{\zeta}_1)(\zeta - \hat{\zeta}_2)\cdots(\zeta - \hat{\zeta}_N)$, the evolution equations are similar in form to (A4.17), (A4.23), (A4.25). For each real $\hat{\zeta}_j, j = 1, \ldots, N$, it is necessary that $b(\hat{\zeta}_j) = \bar{b}(\hat{\zeta}_j) = 0$. If Ω_2 has multiple zeros, the evolution equations involve ζ derivatives of the quantities satisfying the generalized Bloch equations.

In general,

$$\Omega(\zeta) = M(\zeta) + \frac{\Omega_1(\zeta)}{\Omega_2(\zeta)}, \tag{A4.26}$$

where the degree of $\Omega_1(\zeta)$ is less than or equal to the degree of $\Omega_2(\zeta)$ and $M(\zeta)$ is analytic. The only modification in the evolution equation is to replace $\sigma_3 u_t$ by $\sigma_3 u_t + 2M(L^+)u$. An additional point which is worth stressing is that the appearance of the eigenfunctions (quadratic products) in the evolution equations suggests that the eigenfunctions themselves play a well defined role in the physical formulation of the problem. As already demonstrated, the Maxwell-Bloch equations provide one example.

Finally, we will show why the requirement that $b(\hat{\zeta}_j) = \bar{b}(\hat{\zeta}_j) = 0$ is also crucial to the application of IST. Using (3.36) we may compute $I(\psi, \bar{\psi})$:

$$\Omega_2(\zeta)I(\psi, \bar{\psi}) = \int_{-\infty}^{\infty} \sigma_3 u_t \cdot \Omega_2(\zeta)\tilde{\psi}\, dx, \tag{A4.27}$$

where

$$\tilde{\psi} = \begin{pmatrix} \psi_1\bar{\psi}_1 \\ \psi_2\bar{\psi}_2 \end{pmatrix}.$$

Using $L\tilde{\psi} = \zeta\tilde{\psi} + i/2\begin{pmatrix} q \\ -r \end{pmatrix}$, we find

$$\Omega_2(\zeta)I(\psi, \bar{\psi}) = \int_{-\infty}^{\infty} \sigma_3 u_t \cdot \Omega_2(L)\tilde{\psi}\, dx + \frac{\partial}{\partial t}G(\zeta, t) \tag{A4.28}$$

where $G(\zeta, t)$ is a power series in ζ whose coefficients are functionals of the integrated densities $\{C_n\}_{n=1}^{\infty}$. If $\Omega_2(\zeta)$ has N zeros, $\hat{\zeta}_j, j = 1, \ldots, N$, in the complex plane, then $G(\zeta, t)$ is a finite power series of degree (N-1) whose coefficients involve $C_1, \ldots, C_N$, the first N integrated densities. Continuing with the computation, we find on using (3.36) that

$$\Omega_2(\zeta)I(\psi, \bar{\psi}) = -2\Omega_1(\zeta)b\bar{b} + \frac{\partial G}{\partial t}.$$

Thus, from (3.10) (recall we have chosen $A_- = \Omega$),

$$\begin{aligned} A_+ - A_- &= -\frac{1}{\Omega_2(\zeta)}\frac{\partial G}{\partial t}, \\ B_+ &= 0, \\ C_+ &= 0. \end{aligned} \tag{A4.29}$$

In particular, we find on using (A4.29) and integrating (3.7)

$$a(\zeta, t) = a(\zeta, 0) \exp\left\{-\frac{1}{\Omega_2(\zeta)}(G(\zeta, t) - G(\zeta, 0))\right\},$$

$$\bar{a}(\zeta, t) = \bar{a}(\zeta, 0) \exp\left\{\frac{1}{\Omega_2(\zeta)}(G(\zeta, t) - G(\zeta, 0))\right\}.$$

Thus, for $t > 0$, the functions $a(\zeta, t)$ and $\bar{a}(\zeta, t)$ cannot be analytically extended into the upper and lower half ζ-planes respectively, a step crucial to the successful implementation of the inverse method. If $a(\zeta, t)$ were not analytic in the upper half plane then $\int_{-\infty}^{\infty} |q|\, dx$ and $\int_{-\infty}^{\infty} |r|\, dx$ would not be bounded. Alternatively either (a) each zero of $\Omega_2(\zeta)$ is an eigenvalue of the original problem whence $a(\zeta, 0)$ or $\bar{a}(\zeta, 0)$ is zero or (b) $G(\zeta, t) = G(\zeta, 0)$ at the zeros of $\Omega_2(\zeta)$. We have seen that the first alternative leads to a nonunique specification of $\sigma_3 u_t$ (if Im $\hat{\zeta}_j \gtrless 0$) and must be discarded. Therefore, in most cases we must choose the second alternative which means that the first N integrated densities (assuming $\Omega_2(\zeta)$ has N zeros) are motion invariants. As we have noted, this is guaranteed by the vanishing of $b(\hat{\zeta}_j)$ and $\bar{b}(\hat{\zeta}_j)$ for real $\hat{\zeta}_j$, and is automatic for complex $\hat{\zeta}_j$.

Appendix 5. The primordial scattering data

Inspection of (4.43) immediately shows that for the inversion of the scattering data, only certain subsets are required, which we shall call the "primordial" scattering data. We shall now show that all of the scattering data obtainable from the direct scattering problem can be given in terms of this primordial scattering data. Consequently, for any inversion procedure, only these smaller subsets need to be specified and determined.

For the primordial scattering data, we have two possible sets, depending on whether we invert about $x = +\infty$ or $x = -\infty$. Designating these two possible sets by $S_\pm$, we define

$$S_+ \equiv \begin{cases} [\rho(\xi), \bar{\rho}(\xi); \xi = \text{real}] \\ [\rho_k, \zeta_k]_{k=1}^N, [\bar{\rho}_k, \bar{\zeta}_k]_{k=1}^N \end{cases}. \tag{A5.1a}$$

and

$$S_- \equiv \begin{cases} [\sigma(\xi), \bar{\sigma}(\xi); \xi = \text{real}] \\ [\sigma_k, \zeta_k]_{k=1}^N, [\bar{\sigma}_k, \bar{\zeta}_k]_{k=1}^N \end{cases}. \tag{A5.1b}$$

where

$$\rho(\zeta) \equiv \frac{b(\zeta)}{a(\zeta)}, \tag{A5.2a}$$

$$\bar{\rho}(\zeta) \equiv \frac{\bar{b}(\zeta)}{\bar{a}(\zeta)}, \tag{A5.2b}$$

$$\sigma(\zeta) \equiv \frac{\bar{b}(\zeta)}{a(\zeta)}, \tag{A5.2c}$$

$$\bar{\sigma}(\zeta) \equiv \frac{b(\zeta)}{\bar{a}(\zeta)}, \tag{A5.2d}$$

and $\rho_k(\sigma_k)$ is the residue of $\rho(\sigma)$ at $\zeta = \zeta_k$ and $\bar{\rho}_k(\bar{\sigma}_k)$ is the residue of $\bar{\rho}(\bar{\sigma})$ at $\zeta = \bar{\zeta}_k$. Thus

$$\rho_k = \frac{b_k}{a_k'}, \tag{A5.3a}$$

$$\bar{\rho}_k = \frac{\bar{b}_k}{\bar{a}_k'}, \tag{A5.3b}$$

$$\sigma_k = \frac{1}{b_k a_k'}, \tag{A5.3c}$$

$$\bar{\sigma}_k = \frac{1}{\bar{b}_k \bar{a}_k'}, \tag{A5.3d}$$

since at a zero of a or $\bar{a}$, $\bar{b}(\zeta)b(\zeta) = 1$. We are assuming all zeros of $a(\bar{a})$ in the upper (lower) half plane are simple, N and $\bar{N}$ are finite, and that no zeros of a or $\bar{a}$ exist on the real axis. The extension of (A5.1) to these other cases for N and $\bar{N}$ finite is straightforward.

First, we will construct $a(\zeta)$ ($\bar{a}(\zeta)$) in the upper (lower) half plane from either S_+ or S_-. Define the functions

$$f(\zeta) \equiv a(\zeta) \prod_{k=1}^{N} \left(\frac{\zeta - \zeta_k^*}{\zeta - \zeta_k}\right), \tag{A5.4a}$$

$$\bar{f}(\zeta) \equiv \bar{a}(\zeta) \prod_{k=1}^{\bar{N}} \left(\frac{\zeta - \bar{\zeta}_k^*}{\zeta - \bar{\zeta}_k}\right). \tag{A5.4b}$$

Clearly, $\ln f(\zeta)$ ($\ln \bar{f}(\zeta)$) is an analytic function in the upper (lower) half ζ-plane and vanishes at least as fast as ζ^{-1} as $|\zeta| \to \infty$. Thus

$$\oint \frac{d\zeta'}{\zeta' - \zeta} \ln f(\zeta') = 0, \tag{A5.5a}$$

$$\oint \frac{d\zeta'}{\zeta' - \zeta} \ln \bar{f}(\zeta') = 0, \tag{A5.5b}$$

for ζ lying between C and $\bar{C}$ (see Section IV for the definition of these contours). Distorting each contour into the real axis and using (4.4) gives

$$\ln f(\zeta) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta} \Bigg\{ \ln[1 + \rho(\xi')\bar{\rho}(\xi')] + \sum_{k=1}^{N} \ln\left(\frac{\xi' - \zeta_k}{\xi' - \zeta_k^*}\right) + \sum_{k=1}^{\bar{N}} \ln\left(\frac{\xi' - \bar{\zeta}_k}{\xi' - \bar{\zeta}_k^*}\right) \Bigg\}, \tag{A5.6a}$$

when ζ is in the upper half plane, and

$$\ln \bar{f}(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta} \Bigg\{ \ln[1 + \rho(\xi')\bar{\rho}(\xi')] + \sum_{k=1}^{N} \ln\left(\frac{\xi' - \zeta_k}{\xi' - \zeta_k^*}\right) + \sum_{k=1}^{\bar{N}} \ln\left(\frac{\xi' - \bar{\zeta}_k}{\xi' - \bar{\zeta}_k^*}\right) \Bigg\}, \tag{A5.6b}$$

for ζ in the lower half plane we must stipulate the change in $\ln(1 + \rho(\xi')\bar{\rho}(\xi'))$ to be $2\pi i(\bar{N} - N)$ as ξ' goes from $-\infty$ to $+\infty$. Thus, given either S_+ or S_-, [note that $\rho\bar{\rho} = \sigma\bar{\sigma}$], a and $\bar{a}$ in their respective half planes (including the real axis) are determined from (A5.4, 6).

When the potentials are on compact support, the values of b and $\bar{b}$ are also determined in the entire plane by S_+ or S_-. Define $l_\pm$ such that

$$\left.\begin{aligned} r(x) &= 0 \\ q(x) &= 0 \end{aligned}\right\} \text{if } x < l_- \quad \text{or } x > l_+. \tag{A5.7}$$

Then directly from (4.30–33), we have the following nontrivial relations:

$$\oint \frac{d\zeta'}{\zeta' - \zeta}[\rho(\zeta')\, e^{2i\zeta' l_+}]\, e^{2i\zeta'\varepsilon} = 0, \tag{A5.8}$$

$$\oint \frac{d\zeta'}{\zeta' - \zeta}[\bar{\rho}(\zeta')\, e^{-2i\zeta' l_+}]\, e^{-2i\zeta'\varepsilon} = 0, \tag{A5.9}$$

$$\oint \frac{d\zeta'}{\zeta' - \zeta}[\sigma(\zeta')\, e^{-2i\zeta' l_-}]\, e^{2i\zeta'\varepsilon} = 0, \tag{A5.10}$$

$$\oint \frac{d\zeta'}{\zeta' - \zeta}[\bar{\sigma}(\zeta')\, e^{2i\zeta' l_-}]\, e^{-2i\zeta'\varepsilon} = 0, \tag{A5.11}$$

for ζ lying between C and $\bar{C}$, and where ε is any positive number. As a consequence of (A5.8–11), we have from (4.40–41), upon differentiating with respect to ε,

$$\left.\begin{aligned} F(z) &= 0 \\ \bar{F}(z) &= 0 \end{aligned}\right\} \quad \forall z > 2l_+, \tag{A5.12}$$

and

$$\left.\begin{aligned} G(z) &= 0 \\ \bar{G}(z) &= 0 \end{aligned}\right\} \quad \forall z < 2l_-, \tag{A5.13}$$

as required for compact support of $r(x)$ and $q(x)$.

From (A5.8–11), integral representations for $\rho(\zeta)$ and $\bar{\rho}(\zeta)$ in terms of S_+ and for $\sigma(\zeta)$ and $\bar{\sigma}(\zeta)$ in terms of S_- immediately follow. Providing that $r(x)$ and $q(x)$ are continuous, we may set $\varepsilon = 0$ in (A5.8) and obtain

$$\rho(\zeta) = \sum_{k=1}^{N} \rho_k \frac{e^{2il_+(\zeta_k - \zeta)}}{\zeta - \zeta_k} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta}\rho(\xi')\, e^{2il_+(\xi' - \zeta)}, \tag{A5.14a}$$

for ζ in the upper half plane. Similarly from (A5.9) we obtain

$$\bar{\rho}(\zeta) = \sum_{k=1}^{\bar{N}} \bar{\rho}_k \frac{e^{-2il_+(\bar{\zeta}_k - \zeta)}}{\zeta - \bar{\zeta}_k} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta}\rho(\xi')\, e^{-2il_+(\xi' - \zeta)}, \tag{A5.14b}$$

for ζ in the lower half plane. Now, from (A5.2, 4, 6, 14), b and $\bar{b}$ are determined in the respective half planes. Note that (A5.14) requires compact support only at the upper end. To extend b into the lower half plane and $\bar{b}$ into the upper half plane, we use (A5.10, 11) when we have compact support at the lower end. These give

$$\sigma(\zeta) = \sum_{k=1}^{N} \sigma_k \frac{e^{-2il_-(\zeta_k - \zeta)}}{(\zeta - \zeta_k)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta}\sigma(\xi')\, e^{-2il_-(\xi' - \zeta)}, \tag{A5.15a}$$

$$\bar{\sigma}(\zeta) = \sum_{k=1}^{\bar{N}} \bar{\sigma}_k \frac{e^{2il_-(\bar{\zeta}_k - \zeta)}}{(\zeta - \bar{\zeta}_k)} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \zeta}\bar{\sigma}(\xi')\, e^{2il_-(\xi' - \zeta)}, \tag{A5.15b}$$

in each respective half plane.

Appendix 6. *Completeness of the Zakharov-Shabat eigenstates*

Since this eigenvalue problem (2.1) is not self-adjoint in general, one has to use care in obtaining the completeness relation. The first possible difficulty occurs with the natural inner product for this eigenvalue problem, which is blatantly neither positive or negative definite. Consequently, a zero norm need not imply that a vector is zero. Nevertheless, we do find that this natural norm is nonsingular, which does allow us to distinguish between zero and nonzero. First, we will define the natural inner product for this problem and the associated adjoint eigenstates. Then, assuming only simple zeros of a and $\bar{a}$, a finite number of zeros, and no zeros on the real axis, we shall determine all possible inner products between the various eigenstates. Next, assuming completeness, we shall construct what should be the resolution of the identity operator. Now, we note that when the potentials are on compact support, this resolution can be given as the difference of two contour integrals in the complex ζ-plane, and that this result is valid even when the zeros of a and $\bar{a}$ are not simple and even if they lie on the real axis. Finally, we shall show that this resolution *is* the identity operator, thereby proving completeness.

Modifying the Wroskian relations (3.4), we have from (2.1)

$$\frac{d}{dx}W[u(\zeta'), v(\zeta)] = i(\zeta - \zeta')[u_2(\zeta')v_1(\zeta) + u_1(\zeta')v_2(\zeta)], \tag{A6.1}$$

which suggests the natural inner product

$$\langle u(\zeta')|v(\zeta)\rangle \equiv \lim_{L\to+\infty}\int_{-L}^{L} [u_2(\zeta')v_1(\zeta) + u_1(\zeta')v_2(\zeta)]\, dx. \tag{A6.2}$$

We take our basis to be $\phi(\xi)$ and $\bar{\phi}(\xi)$ when $\zeta = \xi =$ real, ϕ_k $(k = 1, 2, \ldots, N)$, and $\bar{\phi}_k$ $(k = 1, 2, \ldots, \bar{N})$ for the bound states. For simplification of the inner products, it becomes convenient to define the adjoint states to be the following *row* matrices:

$$\phi(\xi, x)^A \equiv (\psi_2(\xi, x), \psi_1(\xi, x)), \tag{A6.3a}$$

$$\bar{\phi}(\xi, x)^A \equiv (\bar{\psi}_2(\xi, x), \bar{\psi}_1(\xi, x)), \tag{A6.3b}$$

$$\phi_k(x)^A \equiv (\psi_2(\zeta_k, x), \psi_1(\zeta_k, x)), \tag{A6.3c}$$

$$\bar{\phi}_k(x)^A \equiv (\bar{\psi}_2(\bar{\zeta}_k, x), \bar{\psi}_1(\bar{\zeta}_k, x)). \tag{A6.3d}$$

which satisfy

$$u_{1x}^A - i\zeta u_1^A = r u_2^A, \tag{A6.4a}$$

$$u_{2x}^A + i\zeta u_2^A = q u_1^A. \tag{A6.4b}$$

Then (A6.2) becomes

$$\langle u(\zeta')|v(\zeta)\rangle = \lim_{L\to+\infty}\int_{-L}^{L} u(\zeta', x)^A \cdot v(\zeta, x)\, dx. \tag{A6.5a}$$

Directly from (3.1, 2, 3), (4.19, 20), (A6.1, 3, 4) and upon using the well-known result

$$\lim_{L\to\infty} P\left(\frac{e^{\pm i\alpha L}}{i\alpha}\right) = \pm\pi\delta(\alpha) \tag{A6.5b}$$

we find

$$\langle\phi(\xi')|\phi(\xi)\rangle = 2\pi a\delta(\xi' - \xi), \tag{A6.6a}$$

$$\langle\bar{\phi}(\xi')|\phi(\xi)\rangle = 0, \tag{A6.6b}$$

$$\langle\phi(\xi')|\bar{\phi}(\xi)\rangle = 0, \tag{A6.6c}$$

$$\langle\bar{\phi}(\xi')|\bar{\phi}(\xi)\rangle = -2\pi\bar{a}\delta(\xi' - \xi), \tag{A6.6d}$$

$$\langle\phi_k|\phi_l\rangle = i\delta_l^k a_k', \tag{A6.6e}$$

$$\langle\bar{\phi}_k|\bar{\phi}_l\rangle = i\delta_l^k \bar{a}_k', \tag{A6.6f}$$

with all other inner products being zero. As long as a and $\bar{a}$ are nonzero on the real axis, and have only simple zeros (so that $a_k' \neq 0$ and $\bar{a}_k' \neq 0$), the resulting metric is nonsingular and may be inverted. Consequently, there is a class of functions (column vectors) which can be expanded in this basis. The question is, is this class L_2? For the moment, let us assume that it is L_2, and see what the resolution of the identity operator would be if this were the case. From (A6.6) we would then have

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\delta(x-y) = -i\sum_{k=1}^{N}\phi_k(x)\frac{1}{a_k'}\phi_k(y)^A - i\sum_{k=1}^{\bar{N}}\bar{\phi}_k(x)\frac{1}{\bar{a}_k'}\bar{\phi}_k(y)^A$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\left\{\phi(\xi, x)\frac{1}{a(\xi)}\phi(\xi, y)^A - \bar{\phi}(\xi, x)\frac{1}{\bar{a}(\xi)}\bar{\phi}(\xi, y)^A\right\}. \tag{A6.7}$$

When the potentials are on compact support, (A6.7) can be given by

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\delta(x-y) = \frac{1}{2\pi}\oint_C \frac{d\zeta}{a(\zeta)}\phi(\zeta, x)\phi(\zeta, y)^A$$
$$-\frac{1}{2\pi}\oint_{\bar{C}}\frac{d\zeta}{\bar{a}(\zeta)}\bar{\phi}(\zeta, x)\bar{\phi}(\zeta, y)^A, \tag{A6.8}$$

where the contours C and $\bar{C}$ were defined in Section IV with C passing above all zeros of a and $\bar{C}$ passing below all zeros of $\bar{a}$. We now note that (A6.8) contains also the special cases of a and $\bar{a}$ having nonsimple zeros as well as zeros on the real axis. For the first case, allow two zeros to approach each other to get a double zero, then reduce (A6.8) to integrals along the real axis plus all contributions from the poles. In the second case, let one of the zeros approach the real axis and then reduce (A6.8) to principle value integrals plus all contributions from the poles.

Let us now see if we can evaluate the right hand side of (A6.8) by independent means. The contours in (A6.8) are very suggestive as to how one can do this. First, we use (4.1) and (A6.3) to reduce the right hand side of (A6.8) to

$$\begin{aligned}\text{R.H.S.} = &\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi[\bar{\psi}(\xi,x)\psi(\xi,y)^T + \psi(\xi,x)\bar{\psi}(\xi,y)^T]\sigma_1\\ &+\frac{1}{2\pi}\oint d\zeta\,\frac{b(\zeta)}{a(\zeta)}\psi(\zeta,x)\psi(\zeta,y)^T\sigma_1\\ &-\frac{1}{2\pi}\oint d\zeta\,\frac{\bar{b}(\zeta)}{\bar{a}(\zeta)}\bar{\psi}(\zeta,x)\bar{\psi}(\zeta,y)^T\sigma_1,\end{aligned} \tag{A6.9}$$

where T designates the transpose and

$$\sigma_1 \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{A6.10}$$

Now, we have already shown that ψ and $\bar{\psi}$ under suitable conditions, (4.11), can be given by (4.34, 35) and that $K(x, s)$ and $\bar{K}(x, s)$ exist and are unique. Therefore (A6.9) can be reduced to

$$\begin{aligned}\text{R.H.S.} = &\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\delta(x-y) + \left[\begin{pmatrix}1\\0\end{pmatrix}K(y,x)^T + \begin{pmatrix}0\\1\end{pmatrix}\bar{K}(y,x)^T\right]\sigma_1\theta(x-y)\\ &+[K(x,y)(1,0)+\bar{K}(x,y)(0,1)]\sigma_1\theta(y-x)\\ &+\int_x^\infty ds\int_y^\infty dt\,\delta(t-s)\,[\bar{K}(x,s)K(y,t)^T + K(x,s)\bar{K}(x,t)^T]\sigma_1\\ &+\begin{pmatrix}0\\1\end{pmatrix}(0,1)\sigma_1F(x+y) - \begin{pmatrix}1\\0\end{pmatrix}(1,0)\sigma_1\bar{F}(x+y)\\ &+\int_x^\infty ds[K(x,s)^TF(s+y)(0,1) - \bar{K}(x,s)^T\bar{F}(s+y)(1,0)]\sigma_1\\ &+\int_y^\infty ds\left[\begin{pmatrix}0\\1\end{pmatrix}K(y,s)^TF(s+x) - \begin{pmatrix}1\\0\end{pmatrix}\bar{K}(y,s)^T\bar{F}(x+s)\right]\sigma_1\\ &+\int_x^\infty ds\int_y^\infty dt\,[K(x,s)K(y,t)^TF(s+t) - \bar{K}(x,s)\bar{K}(y,t)^T\bar{F}(s+t)]\sigma_1,\end{aligned} \tag{A6.11}$$

where $\theta(x)$ is unity if $x > 0$ and zero if $x < 0$. By virtue of (4.30, 31), when (4.11) is true, K and $\bar{K}$ *must* satisfy (4.38). (Note that uniqueness is not necessary.) Then using (4.38), and remembering that (4.38) is only valid if $y \geq x$, we find

$$\text{R.H.S.} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\delta(x-y), \tag{A6.12}$$

thereby proving completeness with respect to L_2.

In the next appendix, we will need the limits of (A6.7) when $y \to \pm\infty$, and these are

$$\int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{e^{i\xi y}}{a(\xi)} \left\{ \phi(\xi, x)[1, 0] - \frac{e^{-i\xi y}}{\bar{a}(\xi)} \bar{\phi}(\xi, x)[0, 1] \right\}$$
$$- i \sum_{k=1}^{N} \frac{e^{i\zeta_k y}}{a_k'} \phi_k(x)[1, 0] - i \sum_{k=1}^{\bar{N}} \frac{e^{-i\bar{\zeta}_k y}}{\bar{a}_k'} \bar{\phi}_k(x)[0, 1]$$
$$\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y) \quad \text{as } y \to +\infty, \tag{A6.13}$$

and

$$\int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi y}}{a(\xi)} \psi(\xi, x)[0, 1] + \frac{e^{i\xi y}}{\bar{a}(\xi)} \bar{\psi}(\xi, x)[1, 0]$$
$$- i \sum_{k=1}^{N} \frac{e^{-i\zeta_k y}}{a_k'} \psi_k(x)[0, 1] + i \sum_{k=1}^{N} \frac{e^{i\bar{\zeta}_k y}}{\bar{a}_k'} \bar{\psi}_k(x)[1, 0]$$
$$\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y) \quad \text{as } y \to -\infty. \tag{A6.14}$$

Appendix 7. The Marchenko equations rederived

A completeness relation such as (A6.8) has imbedded in it all of the analytical properties derived in Section IV. Consequently, one can start with the completeness relation and derive the Marchenko equations (4.38, 39). Of course, this is not an independent derivation since, as we saw in Appendix 6, (A6.8) is a consequence of the Marchenko equations. However, this derivation is instructive in that it illustrates how these equations arise from considering a gedenken scattering experiment.

Consider the equations:

$$V_{1x} - V_{1\tau} = qV_2, \tag{A7.1a}$$

$$V_{2x} + V_{2\tau} = rV_1, \tag{A7.1b}$$

where the subscript τ indicate differentiation with respect to the variable τ (not t), and q and r are to be functions only of x and t, but independent of the parameter τ. Note that since t does not enter directly into (A7.1), we may consider it to be fixed. Considering the $x - \tau$ space, we see that (A7.1) describes the evolution of V with respect to τ, so we can consider pulses in the $x - \tau$ space, being "scattered" by the "stationary" (with respect to τ) potentials r and q. Using separation of variables to solve (A7.1), the general solution will be

$$V = \int_{-\infty}^{\infty} d\xi \, e^{-i\xi\tau} v(\xi, x) + \sum_{k=1}^{N} v_k(x) \, e^{-i\zeta_k \tau} + \sum_{k=1}^{\bar{N}} \bar{v}_k(x) \, e^{-i\bar{\zeta}_k \tau}, \tag{A7.2}$$

where v, v_k and $\bar{v}_k$ are arbitrary eigenstates of (2.1).

Let's first consider a delta function pulse of the form

$$V(\tau \to -\infty, x) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta(x - \tau). \tag{A7.3}$$

Clearly, from (A6.14) the required solution is

$$V(\tau, x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi\tau}}{a(\xi)} \psi(\xi, x) - i \sum_{k=1}^{N} \frac{e^{-i\zeta_k\tau}}{a'_k} \psi_k(x). \tag{A7.4}$$

Furthermore, since (A7.1) possesses characteristics, (A7.3) shows that

$$V(\tau, x) = 0 \quad \text{if } x > \tau. \tag{A7.5}$$

Next, consider the conjugate equations

$$U_{1x} + U_{1\tau} = qU_2, \tag{A7.6a}$$

$$U_{2x} - U_{2\tau} = rU_1. \tag{A7.6b}$$

If we consider the initial pulse

$$U(\tau \to -\infty, x) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta(x - \tau), \tag{A7.7}$$

then the exact solution is similarly

$$U(\tau, x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{e^{i\xi\tau}}{\bar{a}(\xi)} \bar{\psi}(\xi, x) + i \sum_{k=1}^{\bar{N}} \frac{e^{i\bar{\zeta}_k\tau}}{\bar{a}'_k} \bar{\psi}_k(x), \tag{A7.8}$$

and again

$$U(\tau, x) = 0 \quad \text{if } x > \tau. \tag{A7.9}$$

As each pulse (A7.3, 7) propagates forward in τ, it will eventually feel the potentials r and q and will be scattered. Now, use (4.5) and (4.20a) to express (A7.4) in terms of ϕ and $\bar{\phi}$. Next, replace ϕ and $\bar{\phi}$ with the expressions (4.36, 37) to obtain (A7.4) in terms of L and $\bar{L}$. Then by (4.41a), upon replacing τ by y, we see that this is just (4.39a), upon requiring (A7.5). Similar treatment of (A7.8, 9) will then give (4.39b). If one now sends in pulses from the right instead of from the left, similar use of (A6.13) will give the Marchenko equations for the right end (4.38).

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