Remark 2 Integrals of motion for NLS - rigorous approach

(R1) $\partial_t \underline{\sigma} = U_2 \underline{\sigma}$; (R2) $\partial_x \underline{\sigma} = U_1 \underline{\sigma}$

Just for this part, assume that $x \in [-a, a]$ with periodic BC's imposed on P, Q and $\underline{\sigma}$.

(R2) gives $\underline{\sigma}_t(y) = T_t(y, x, k)\, \underline{\sigma}_t(x)$,

where

$$(R3) \quad \begin{cases} \partial_y T(y, x, k) = U_1(y, k, t)\, T_t(y, x, k) \\ T_t(x, x, k) = Id \end{cases}$$

T is the transition matrix

Then $T_t(y, x, k) = P\exp \int_x^y U_1(z, k, t)\, dz$

$$\left(= \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y dz_1 \ldots \int_x^y dz_n\, P\left(U_1(z_1, k, t) \ldots U_1(z_n, k, t)\right)\right)$$

P is the path-ordering operator,

$$P\left(U(z_1, \lambda)\, U(z_2, \lambda)\right) = \begin{cases} U_1(z_1, \lambda)\, U_1(z_2, \lambda) & z_1 < z_2 \\ U_1(z_2, \lambda)\, U_1(z_1, \lambda) & z_2 > z_1 \end{cases}$$

Time evolution of the transition matrix:

$$\partial_t T_t(y,x,k) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y dz_1 \dots \int_x^y dz_n$$

$$\sum_{m=1}^{n} P\Big(U_1(z_1,k,t) \dots \underbrace{\dot{U}_1(z_m,k,t)}_{\partial U_2(z_m,k,t) - [U_1(z_m,k,t), U_2(z_m,k,t)]} \dots U_n(z_n,k,t) \Big)$$

$\to$ 0 (under the sign of P-ordering)

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{n} \prod_{\substack{\ell=1 \\ \ell \neq m}}^{n} \int_x^z dz_\ell \; P\Big(U_1(z_1) \dots \underbrace{\big(U_1(y) - U_1(x)\big)}_{m\text{-th term}} \dots$$

$$U_n(z_n) \Big) = U_2(y,k,t)\, T_t(y,x,k) - T_t(y,x,k)\, U_2(y,k,t) \qquad (R4)$$

Define the monodromy matrix,

$$T_t(a,k) = T_t(a,-a,k) \qquad (R5)$$

(R4) means

$$(R6) \qquad \partial_t T_t(a,k) \underset{\substack{\uparrow \\ \text{Periodic} \\ \text{BC}}}{=} [U_2(a,k,t), T_t(a,k)]$$

The expansion of $\mathrm{tr}\, T_t(a,k)$ in powers of k gives all the integrals of motion:

really, taking the trace (in the space of 2×2 matrices) of (R6) one finds

$$\partial_t \operatorname{tr} T_t(q,k) = 0$$

Deriving the explicit expressions for integrals of motion is actually quite involved and is based on the analysis of the following matrix R-equation:

$$T_t(x,y,k) = \left(\mathbb{I} + \underbrace{W(x,t,\lambda)}_{\text{Anti-diagonal}}\right) e^{\overbrace{Z(x,y,t,\lambda)}^{\text{Diagonal}}} \times \left(\mathbb{I} + W(x,t,\lambda)\right)^{-1}$$

$$\downarrow$$

$$\frac{\partial W}{\partial x} + [W, U_D] + W U_A W - U_A = 0,$$

where $U_1 = \underbrace{U_D}_{\text{Diag}} + \underbrace{U_A}_{\text{Anti-diag.}}$

$\boxed{\text{Solving the linear system}}$

Let (6) $\begin{cases} \psi = e^{k^2 t/2}\,\phi \\ \bar{\psi} = e^{-k^2 t/2}\,\bar{\phi} \end{cases}$ be two linearly-indep. solutions to the linear problem (3)

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

(6') $\begin{cases} \phi \simeq \begin{pmatrix} e^{-ikx/2} \\ 0 \end{pmatrix}, & x \to -\infty \\ \bar{\phi} \simeq \begin{pmatrix} 0 \\ -e^{ikx/2} \end{pmatrix}, & x \to -\infty \end{cases}$

(6), (6') is well-motivated:

For $|x| \to \infty$

$$(*) \quad \partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -ik/2 & 0 \\ 0 & ik/2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$(**) \quad \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} k^2/2 & 0 \\ 0 & -k^2/2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\Rightarrow \quad \psi = \begin{pmatrix} c_1 e^{-i\frac{k}{2}x + \frac{k^2}{2}t} \\ c_2 e^{i\frac{k}{2}x - \frac{k^2}{2}t} \end{pmatrix}$$

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Therefore, (6)

$$(7)\quad \begin{cases} \phi(x,t) \simeq \begin{pmatrix} a(k,t)e^{-ikx/2} \\ b(k,t)e^{ikx/2} \end{pmatrix} \\ \bar{\phi}(x,t) \simeq \begin{pmatrix} \tilde{b}(k,t)e^{-ikx/2} \\ -\tilde{a}(k,t)e^{ikx/2} \end{pmatrix} \end{cases}, \quad x \to +\infty$$

$a, b, \tilde{a}, \tilde{b}$ are "scattering amplitudes".

Using $(**)$ and (6):

$$+\frac{k^2}{2}\begin{pmatrix} \cancel{ae^{-ikx/2}} \\ be^{ikx/2} \end{pmatrix} + \begin{pmatrix} \dot{a}\, e^{-ikx/2} \\ \dot{b}\, e^{ikx/2} \end{pmatrix} = \begin{pmatrix} \frac{k^2}{2}\cancel{a e^{-ikx/2}} \\ -\frac{k^2}{2} b e^{ikx/2} \end{pmatrix}$$

$$-\frac{k^2}{2}\begin{pmatrix} \tilde{b}e^{-ik\frac{x}{2}} \\ \cancel{-\tilde{a}e^{ik\frac{x}{2}}} \end{pmatrix} + \begin{pmatrix} \dot{\tilde{b}}\, e^{-ikx/2} \\ -\dot{\tilde{a}}\, e^{ikx/2} \end{pmatrix} = \begin{pmatrix} \frac{k^2}{2}\tilde{b}e^{-ikx/2} \\ +\frac{k^2}{2}\cancel{\tilde{a} e^{ikx/2}} \end{pmatrix}$$

$$\Rightarrow \quad (8)\quad \begin{cases} a(k,t) = a_0(k) \\ b(k,t) = e^{-k^2 t}\, b_0(k) \\ \tilde{a}(k,t) = \tilde{a}_0(k) \\ \tilde{b}(k,t) = e^{k^2 t}\, \tilde{b}_0(k) \end{cases}$$

Moreover, define

$$W = \phi_1 \bar{\Phi}_2 - \phi_2 \bar{\Phi}_1 \quad (\text{"Wronskian"})$$

$$(= \varepsilon_{ij} \phi_i \bar{\Phi}_j)$$

$$\partial_t W = \varepsilon_{ij} (\dot{\phi}_i \bar{\Phi}_j + \phi_i \dot{\bar{\Phi}}_j)$$

Are we doing it for ϕ's or ψ's?

$$= \varepsilon_{ij} U_{2ik} \phi_k \bar{\Phi}_j + \varepsilon_{ij} \phi_i U_{2jk} \bar{\Phi}_k$$

$$= \varepsilon_{ij} U_{2ik} \phi_k \bar{\Phi}_j - \varepsilon_{ij} \phi_j U_{2ik} \bar{\Phi}_k$$

$$= \varepsilon_{ij} U_{2ik} \phi_k \bar{\Phi}_j - \varepsilon_{ik} \phi_k U_{2ij} \bar{\Phi}_j$$

$$= \underbrace{(\varepsilon_{ij} U_{2ik} - \varepsilon_{ik} U_{2ij})}_{\text{skew-sym}} \phi_k \bar{\Phi}_j$$

$$= (\varepsilon_{i2} U_{2i1} - \varepsilon_{i1} U_{2i2}) W$$

$$= (U_{211} + U_{222}) W = (\operatorname{tr} U_2) W = 0$$

Similarly, $\partial_x W = (\operatorname{Tr} U_1) W = 0$

Therefore $W : (x,t) \mapsto W(x,t)$ is constant.

So $\lim_{x \to +\infty} W(x,t) = \lim_{x \to -\infty} W(x,t)$

$= -1 \Rightarrow$

$$(9) \quad \tilde{a}_0(k)\, a_0(k) + \tilde{b}_0(k)\, b_0(k) = 1$$

(The authors are not using subscript $_0$)

Inverse scattering transform: from $a, b, \tilde{a}, \tilde{b}$ to P, Q
(inverse scattering transform):
Define reflection coefficients $r, \tilde{r}$:

$$r(k) := \frac{b(k)}{a(k)} \qquad \tilde{r}(k) := \frac{\tilde{b}(k)}{g\,\tilde{a}(k)}$$

Coupling constant

Define A_t, B_t:

$$(10) \quad \begin{cases} A_t(x) = \int_{\mathbb{R}} \frac{dk}{2\pi}\, r(k)\, e^{ikx - k^2 t} \\ B_t(x) = \int_{\mathbb{R}} \frac{dk}{2\pi}\, \tilde{r}(k)\, e^{-ikx + k^2 t} \end{cases}$$

Let $\mathcal{A}_{x,t},\ \mathcal{B}_{x,t} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$

be a pair of integral operators with the kernels

$$\begin{cases} \mathcal{A}_{x,t}(\sigma,\sigma') = A_t(x+\sigma+\sigma') \\ \mathcal{B}_{x,t}(\sigma,\sigma') = B_t(x+\sigma+\sigma') \end{cases}$$

N.B. Both kernels and functions obey the heat eq-n in space-time.
Assume A_t, B_t vanish sufficiently fast for $x \to \infty$.

$\langle\delta| \to \langle 0|$? Claim $Q(x,t) = \langle\delta|\, \mathcal{A}_{x,t}(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t})^{-1}|\delta\rangle$

$$(11) \quad P(x,t) = \langle\delta|\, \mathcal{B}_{x,t}(I + g\,\mathcal{A}_{x,t}\mathcal{B}_{x,t})^{-1}|\delta\rangle,$$

where for any integral operator $\mathcal{O}$, $\langle\delta|\mathcal{O}|\delta\rangle \stackrel{\text{I}}{=} \mathcal{O}(0,0)$
Kernel

Moreover,

(12) $$g P Q = \partial_x^2 \log \operatorname{Det}(I + g \mathcal{B}_{x,t} \mathcal{A}_{x,t})$$

WOW Let's check (12) first.

Lemma.

(13) $$\partial_x (\mathcal{B}_{x,t} \mathcal{A}_{x,t}) = - \mathcal{B}_{x,t} |\delta\rangle\langle\delta| \mathcal{A}_{x,t}$$

(13') $$\partial_x (\mathcal{A}_{x,t} \mathcal{B}_{x,t}) = - \mathcal{A}_{x,t} |\delta\rangle\langle\delta| \mathcal{A}_{x,t}$$

\# $$\mathcal{B}_{x,t} \mathcal{A}_{x,t} (\sigma, \sigma')$$
$$= \int_0^\infty d\sigma'' \, \mathcal{B}_{x,t}(\sigma + \sigma'') \, \mathcal{A}_{x,t}(\sigma'' + \sigma')$$
$$= \int_0^\infty d\sigma'' \, B_t(x + \sigma + \sigma'') A_t(x + \sigma'' + \sigma')$$

$$\partial_x \mathcal{B}_{x,t} \mathcal{A}_{x,t} (\sigma, \sigma')$$
$$= \int_0^\infty d\sigma'' \frac{\partial}{\partial \sigma''} \left(B_t(x + \sigma + \sigma'') A_t(x + \sigma'' + \sigma') \right)$$
$$= - B_t(x + \sigma) A_t(x + \sigma') \Rightarrow$$

$$\partial_x (\mathcal{B}_{x,t} \mathcal{A}_{x,t}) f(\sigma)$$
$$= - \underbrace{B_t(x + \sigma)}_{\mathcal{B}_{x,t}(\delta_0)} \underbrace{\int_0^\infty d\sigma' A_t(x + \sigma') f(\sigma')}_{(\mathcal{A}_{x,t} f)(0)}$$

– a projector onto $B_t(x + \sigma)$

In bracket notations,

$$\partial_x B_{x,t} A_{x,t} = -B_{x,t}|\delta\rangle\langle\delta| A_{x,t}$$

The proof for $\partial_x A_{x,t} B_{x,t}$ is exactly the same

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The verification of (12):

$$\partial_x \log \mathrm{Det}(I + g B_{x,t} A_{x,t})$$

$$= \partial_x \mathrm{Tr} \log\left(I + g B_{x,t} A_{x,t}\right)$$

$$= -\partial_x \mathrm{Tr} \sum_{n=1}^{\infty} \frac{(-g)^n}{n} \left(B_{x,t} A_{x,t}\right)^n$$

$$= -\mathrm{Tr} \sum_{n=1}^{\infty} \frac{(-g)^n}{\cancel{n}} \left(B_{x,t} A_{x,t}\right)^{n-1} \partial_x \left(B_{x,t} A_{x,t}\right)$$

Cyclic property of Tr

$$= -\mathrm{Tr} \sum_{n=1}^{\infty} (-g)^{n-1} \left(B_{x,t} A_{x,t}\right)^{n-1} \left(-g B_{x,t} |\delta\rangle\langle\delta| A_{x,t}\right) = +g \,\mathrm{Tr}\, A_{x,t} \left(I + g B_{x,t} A_{x,t}\right)^{-1} B_{x,t} |\delta\rangle\langle\delta|$$

Tr of a projector

$$= +g \langle\delta| A_{x,t} \left(I + g B_{x,t} A_{x,t}\right)^{-1} B_{x,t} |\delta\rangle$$

$$= +g \langle\delta| A_{x,t} \sum_{n=0}^{\infty} (-g)^n \left(B_{x,t} A_{x,t}\right)^n B_{x,t} |\delta\rangle$$

$$= +g \langle\delta| \sum_{n=0}^{\infty} (-g)^n \left(A_{x,t} B_{x,t}\right)^n A_{x,t} B_{x,t} |\delta\rangle$$

$$= +g \langle\delta| \left(I + A_{x,t} B_{x,t}\right)^{-1} A_{x,t} B_{x,t} |\delta\rangle$$

$$= g \langle \delta | \left(I + A_{x,t} B_{x,t} \right)^{-1} - I | \delta \rangle$$ (11)

$$\partial_x^2 \log \mathrm{Det} \left(I + g B_{x,t} A_{x,t} \right)$$

$$= -g \langle \delta | \left(I + A_{x,t} B_{x,t} \right)^{-1} \partial_x \left(A_{x,t} B_{x,t} \right) \left(I + A_{x,t} B_{x,t} \right)^{-1} | \delta \rangle$$

$$= g \langle \delta | \left(I + A_{x,t} B_{x,t} \right)^{-1} A_{x,t} | \delta \rangle \langle \delta | B_{x,t} \left(I + A_{x,t} B_{x,t} \right)^{-1} | \delta \rangle = g P Q$$

The key to the above factorization is the link between $\partial_x \left(A_{x,t} B_{x,t} \right)$ and projectors. It seems very general - we didn't have to use any properties of A_t, B_t.

Verification of (11):

\# Summary of bracket notations:

$$(\mathcal{A}_{xt} f)(\sigma) = \int_0^\infty d\sigma' \, A_t(x+\sigma+\sigma') f(\sigma')$$
$$= \langle \delta_\sigma | \mathcal{A}_{xt} | f \rangle$$

$\langle \delta_\sigma | f \rangle = f(\sigma)$; $|\delta_w\rangle$ is the vector with components $\delta(w-v)$; $|\delta_0\rangle \equiv |\delta\rangle$.

Natural notations: $f(\sigma) = \langle \delta_\sigma | f \rangle = \int dw \, \delta(\sigma - w)$ $f(w) = f(\sigma)$. Tensor products corresponding to finite rank operators are easy to represent: for $\varphi, \psi \in L^2(\mathbb{R}^+)$, $|\varphi\rangle\langle\psi|$ acts as follows: $|f\rangle \longmapsto |\varphi\rangle \underbrace{\langle \psi | f \rangle}_{L^2 \text{ inner product}}$

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It follows from (10) that

$$\begin{cases} \partial_t \mathcal{A}_{xt} = \partial_x^2 \mathcal{A}_{xt} \\ \partial_t \mathcal{B}_{xt} = -\partial_x^2 \mathcal{B}_{x,t} \end{cases}$$

Recall also (13), (13').

The following is straightforward:

$$\partial_t (I + g \mathcal{A}_{x,t} \mathcal{B}_{x,t})^{-1} = -(I + g \mathcal{A}_{x,t} \mathcal{B}_{x,t})^{-1}$$
$$g \, \partial_t (\mathcal{A}_{x,t} \mathcal{B}_{x,t}) (I + g \mathcal{A}_{x,t} \mathcal{B}_{x,t})^{-1}$$

Let's verify the first line of (1):

$$\partial_t Q = \langle\delta|\, \partial_t \mathcal{A}_{x,t} \left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1} |\delta\rangle$$

$$- g\,\langle\delta|\, \mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1} \partial_t\left(\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}$$

$$|\delta\rangle = \langle\delta|\left(\partial_x^2 \mathcal{A}_{x,t}\right)\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}|\delta\rangle$$

$$- g\,\langle\delta|\, \mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}\Big(\underbrace{-\left(\partial_x^2 \mathcal{B}_{x,t}\right)\mathcal{A}_{x,t}}_{(*)}$$

$$\underline{+\ \mathcal{B}_{x,t}\left(\partial_x^2 \mathcal{A}_{x,t}\right)}\Big)\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}|\delta\rangle =$$

$$\#\ (*) = \partial_x\left(-\left(\partial_x \mathcal{B}_{x,t}\right)\mathcal{A}_{x,t} + \mathcal{B}_{x,t}\,\partial_x \mathcal{A}_{x,t}\right)$$

$$= \partial_x \mathcal{B}_{x,t}\,|\delta\rangle\langle\delta|\,\mathcal{A}_{x,t} - \mathcal{B}_{x,t}\,|\delta\rangle\langle\delta|\,\partial_x \mathcal{A}_{x,t}$$

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Apply (B), the kernel $\partial_x \mathcal{B}_{x,t}$ has the same properties as $\mathcal{B}_{x,t}$ for integration by parts

$$= \langle\delta|\, \partial_x^2 \mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}|\delta\rangle$$

$$- g\,\langle\delta|\, \mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}\partial_x \mathcal{B}_{x,t}$$

$$|\delta\rangle\langle\delta|\,\mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}|\delta\rangle$$

$$+ g\,\langle\delta|\,\mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}\mathcal{B}_{x,t}|\delta\rangle$$

$$\langle\delta|\,\partial_x \mathcal{A}_{x,t}\left(I + g\,\mathcal{B}_{x,t}\mathcal{A}_{x,t}\right)^{-1}|\delta\rangle =$$

$$\#\ \langle\delta|\,\mathcal{A}\left(I + g\,\mathcal{B}\mathcal{A}\right)^{-1}\mathcal{B}|\delta\rangle = \langle\delta|\,\mathcal{A}\sum_{k=0}^{\infty}(-g)^k\left(\mathcal{B}\mathcal{A}\right)^k\mathcal{B}|\delta\rangle$$

$$= \langle\delta|\sum_{k=0}^{\infty}(-g)^k\,\underbrace{\mathcal{A}\,\mathcal{B}\,\mathcal{A}\,\mathcal{B}\,\mathcal{A}\ldots\mathcal{B}\,\mathcal{A}}_{k \text{ times}}\mathcal{B}|\delta\rangle = \langle\delta|\sum_{k=0}^{\infty}(-g)^k\left(\mathcal{A}\mathcal{B}\right)^{k+1}|\delta\rangle$$

$$= \langle\delta|\left(I + g\,\mathcal{A}\mathcal{B}\right)^{-1}\mathcal{A}\mathcal{B}|\delta\rangle$$

$$= -\langle\delta|\sum_{k=1}^{\infty}(-g)^{k-1}\left(\mathcal{A}\mathcal{B}\right)^k\rangle = -\frac{1}{g}\langle\delta|\sum_{k=1}^{\infty}(-g)^k\left(\mathcal{A}\mathcal{B}\right)^k|\delta\rangle = -\frac{1}{g}\langle\delta|\left(I + g\,\mathcal{A}\mathcal{B}\right)^{-1} - 1|\delta\rangle$$

#

$$\equiv \langle \delta | (\partial_x^2 \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$- g \langle \delta | \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} (\partial_x B_{xt}) | \delta \rangle Q$$

$$- \langle \delta | ((I + g \mathcal{A}_{xt} B_{xt})^{-1} - I) | \delta \rangle \langle \delta | (\partial_x \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

(Checked to $O(g)$)

$$\partial_x Q = \partial_x \langle \delta | \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$= \langle \delta | (\partial_x \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle - g \langle \delta | \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1}$$

$$\partial_x (B_{xt} \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$\overset{(13)}{=} \langle \delta | (\partial_x \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle + g \langle \delta | \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} B_{xt} | \delta \rangle$$

$$\times \; Q(x,t) = \langle \delta | \partial_x \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

(Checked to $O(g)$)

$$- \langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} - I | \delta \rangle \; Q(x,t)$$

$$\partial_x^2 Q = \langle \delta | \partial_x^2 \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$+ g \langle \delta | \partial_x \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} B_{xt} | \delta \rangle \cdot Q(x,t)$$

$$- g \langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} \mathcal{A}_{xt} | \delta \rangle \langle \delta | B_{xt} (I + g \mathcal{A}_{xt} B_{xt})^{-1} | \delta \rangle Q$$

$$- \langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} - I | \delta \rangle \langle \delta | \partial_x \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$+ \left(\langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} - I | \delta \rangle \right)^2 Q \equiv$$

$$\# \quad \langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} \mathcal{A}_{xt} | \delta \rangle = \langle \delta | \sum_{k=0}^{\infty} (-g)^k \underbrace{\mathcal{A}_{xt} B_{xt} \ldots \mathcal{A}_{xt} B_{xt}}_{k \text{ times}} \mathcal{A}_{xt} | \delta \rangle$$

$$= \langle \delta | \mathcal{A}_{xt} \sum_{k=0}^{\infty} (-g)^k (B_{xt} \mathcal{A}_{xt})^k | \delta \rangle = Q \quad \#$$

$$\equiv \langle \delta | (\partial_x^2 \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle + g \langle \delta | \partial_x \mathcal{A}_{xt} (I + g B_{xt} \mathcal{A}_{xt})^{-1} B_{xt} | \delta \rangle Q$$

$$- g P Q^2 - \langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} - I | \delta \rangle \langle \delta | (\partial_x \mathcal{A}_{xt}) (I + g B_{xt} \mathcal{A}_{xt})^{-1} | \delta \rangle$$

$$+ \left(\langle \delta | (I + g \mathcal{A}_{xt} B_{xt})^{-1} - I | \delta \rangle \right)^2 Q$$

$$\partial_t Q - \partial_x^2 Q =$$

$$-g Q \Big(\langle \delta | \mathcal{A}_{x+} (I + g \mathcal{B}_{x+} \mathcal{A}_{+})^{-1} \partial_x \mathcal{B}_{x+} | \delta \rangle + \langle \delta | \partial_x \mathcal{A}_{x+} (I + g \mathcal{B}_{x+} \mathcal{A}_{+})^{-1} \mathcal{B}_{x+} | \delta \rangle \Big) + g P Q^2$$

$$- \Big(\langle \delta | (I + g \mathcal{A}_{x+} \mathcal{B}_{x+})^{-1} - I | \delta \rangle \Big)^2 Q$$

Gives $2g PQ^2$ up to $O(g^2)$

$$= g P Q^2 - g Q \Big(\langle \delta | (\partial_x \mathcal{A}_{x+}) (I + g \mathcal{B}_{x+} \mathcal{A})^{-1} \mathcal{B}_{x+} | \delta \rangle$$

$$+ \langle \delta | \mathcal{A}_{x+} (I + g \mathcal{B}_{x+} \mathcal{A})^{-1} (\partial_x \mathcal{B}_{x+}) | \delta \rangle$$

$$- \langle \delta | (I + g \mathcal{A}\mathcal{B})^{-1} \mathcal{A}\mathcal{B} | \delta \rangle \langle \delta | (I + g \mathcal{A}\mathcal{B}) \mathcal{A}\mathcal{B} | \delta \rangle \Big)$$

$$= g P Q^2 - g Q \Big(\langle \delta | \partial \mathcal{A} (I + g \mathcal{B}\mathcal{A})^{-1} \mathcal{B} | \delta \rangle$$

$$+ \langle \delta | \mathcal{A} (I + g \mathcal{B}\mathcal{A})^{-1} \partial \mathcal{B} | \delta \rangle$$

$$- \underbrace{\langle \delta | \mathcal{A} (I + g \mathcal{B}\mathcal{A})^{-1} \mathcal{B} | \delta \rangle \langle \delta | \mathcal{A} (I + g \mathcal{A}\mathcal{B})^{-1} \mathcal{B} | \delta \rangle}_{-\langle \delta | \mathcal{A} \partial_x (I + g \mathcal{B}\mathcal{A})^{-1} \mathcal{B} | \delta \rangle} \Big)$$

$$= g P Q^2 - g Q \partial_x \langle \delta | \mathcal{A} (I + g \mathcal{B}\mathcal{A})^{-1} \mathcal{B} | \delta \rangle$$

$$= g P Q^2 - g Q \partial_x \langle \delta | (I + g \mathcal{A}\mathcal{B})^{-1} \mathcal{A}\mathcal{B} | \delta \rangle$$

$$= g P Q^2 + Q \partial_x \langle \delta | (I + g \mathcal{A}\mathcal{B})^{-1} - I | \delta \rangle$$

$$= g P Q^2 + g Q \underbrace{\langle \delta | (I + g \mathcal{A}\mathcal{B})^{-1} \mathcal{A} | \delta \rangle}_{Q} \underbrace{\langle \delta | \mathcal{B} (I + g \mathcal{A}\mathcal{B})^{-1} | \delta \rangle}_{P}$$

$$= 2 g P Q^2$$

Done